# Heat and Mass Transfer

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**Module 1. Basic Concepts, Conductive Heat Transfer and Extended Surfaces**

## Lesson 1. Heat Transfer, Importance of Heat Transfer, modes of Heat Transfer

**Temperature:** Temperature is an intensive property that indicates the thermal state of a system or a body. Temperature is a measure of internal energy possessed by a system and gives the direction in which energy in the form of heat will flow. It is generally denoted by 'T' and following scales are used to measure temperature

- Celsius or Centigrade Scale: According to this scale the freezing point of water is assigned a value of zero and boiling point value is 100. It is represented by $^oC$.
- Kelvin Scale: According to this scale the freezing point of water is assigned a value of 273 and boiling point value is 373. It is represented by K.
- Fahrenheit Scale: According to this scale the freezing point of water is assigned a value of 32 and boiling point value is 212. It is represented by $^oF$.

The relationship between these three scales of temperature measurement is given as:

$$\frac{C}{5} = \frac{F-32}{9} = \frac{K-273}{5} \qquad (1)$$

**Heat:** Heat is a form of energy which is transient in nature and it flows from one point to another point. When two bodies of different temperatures come in contact with each other, the two temperatures approach each other and after some time become equal. This equalization of temperature of the bodies is on account of flow of energy in the form of heat from one body to another. Therefore, heat may be defined as flow of energy from one body to another body by virtue of temperature difference between them. The net flow of energy always occurs from high temperature body towards low temperature body and this flow of heat stops the moment temperature of both the bodies are equal. Thus, flow of heat is a non-mechanical transfer of energy occurring due to a temperature difference between two bodies.

According to the international system of units (S.I.), the unit of measurement of heat is Joule.

1 Joule = 1 Newton meter

= Watt-second

1 kcal = 4.182 X $10^3$ Joule

1 kWh = 3600 kJ

**Heat Transfer and Thermodynamics:**

Thermodynamics and heat transfer are related to each other. The laws of thermodynamics form the basis of science of heat transfer. However, there are few fundamental differences between thermodynamics and heat transfer which are given in Table 1.1:

Table 1 : Fundamental Differences in Thermodynamics and Heat Transfer

| | | |
|---|---|---|
| 1. | Thermodynamics is a science which deals with equilibrium states and the changes from one state to another during a process. | Heat transfer is a non-equilibrium phenomenon as it occurs when thermal equilibrium is disturbed. |
| 2 | Thermodynamics is a science which deals with amount of heat transferred during a process. | Heat transfer is a science that deals with the rate as well as mode of heat transfer during a process. |
| 3. | Thermodynamics is a science which deals with amount of work done during a process. | Heat transfer indicates the temperature distribution inside a body. |

**Modes of Heat Transfer:**

Heat transfer is the study of transmission of thermal energy from a high temperature region / body to a low temperature region / body on account of temperature difference. The rate of heat transfer is directly proportional to the temperature difference between the heat exchanging regions / bodies. Once the process of heat energy is complete, it is stored in one or more forms of energy such as potential, kinetic and internal energy. It is pertinent to mention that energy in transition as heat can never be measured; however, it is determined in terms of observed changes in other forms of energy. Transfer of heat between two regions / bodies maintained at different temperatures can occur in three different modes namely:

- Conduction
- Convection
- Radiation

In the conduction and convection modes, heat flows from high temperature to low temperature region / body whereas in radiation mode, transfer of heat takes place from both the bodies towards each other. However, net transfer of heat is always from high temperature body to low temperature body. Mechanism of heat transfer in each mode is different and controlled by different laws.

**Conduction:**

Conduction is a process of heat transfer from a high temperature region to a low temperature region with in a body or between different bodies which are in direct physical contact. In heat conduction, energy is transferred due to exchange of molecular kinetic energy. According to kinetic theory, temperature of body is proportional to the mean kinetic energy of its constituent molecules. As the temperature in one region of a body increases, kinetic energy of molecules in that region also increases as compared to that of the molecules of adjacent low temperature region. High energy molecules transfer a part of their energy by impact in case of fluids or by diffusion in case of metals to low energy molecules, thereby resulting in increase in their energy levels, hence temperature. Likewise, this process of energy transfer by molecular activity continues till temperature along the entire length of the body becomes equal and has been depicted in Figure 1.

Heat transfer by conduction in solids, liquids and gases is determined by the thermal conductivity and temperature difference. The basic law of heat transfer by conduction was proposed by the French Scientist J. B. J. Fourier in 1822 and one dimensional Conduction rate equation described by the Fourier Law is written as:

$$Q_x = -k A \frac{dT}{dx} \quad (2)$$

Where, $Q_x$ - Heat Flow, (W)

k – Thermal conductivity of the material, ( W/(m-K)

A – Cross-sectional area in the direction of heat flow, ($m^2$)

dT/dx – Temperature gradient, (K/m)

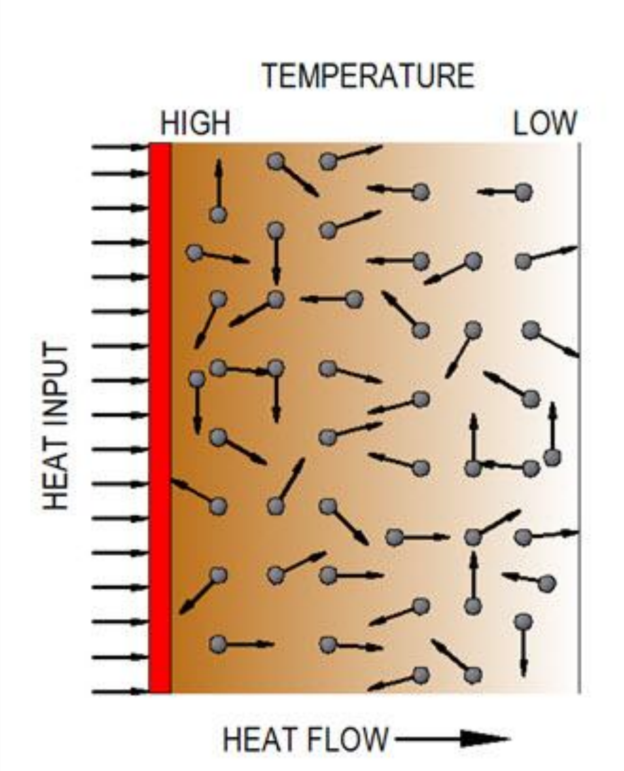



**Figure 1**

**Convection:**

Heat transfer by convection occurs when a fluid (Liquid and gas) comes in contact with a solid through direct contact and a temperature difference exists between them. Heat transfer by Convection occurs under the combined action of heat conduction and mixing motion.

When a fluid comes in contact with a hot surface, energy in form of heat flows by conduction from hot surface to the adjacent stagnant layer of fluid particles, thereby increasing their temperature and internal energy. Due to increase in temperature, density of the fluid particles decreases and they become lighter as compared to the surrounding fluid particles. The lighter fluid particles move up to a region of lower temperature with in the fluid where they mix and exchange a part of their energy with colder fluid particles. Simultaneously, the cold fluid particles move downwards to occupy the space vacated by hot fluid particles. This upward and downward movement of hot and cold fluid particles continues till temperature of the fluid and the surface becomes equal. The convection heat transfer process has been shown in Figure 2.

The upwards movement of hot fluid particles and downward movement of cold fluid particles is called convectional currents. If the convectional currents are set up only due to density differences, then the heat transfer process is termed as natural or free convection.

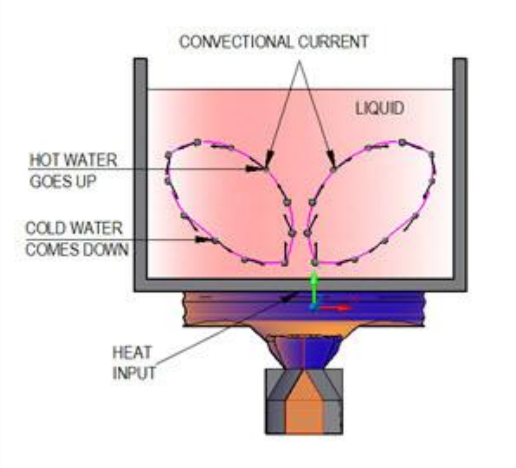



**Figure 2**

However, if the convectional currents are caused by some external means such as blower, fan, pump etc. then heat transfer process is called forced convection.

It is virtually impossible to observe pure heat conduction in a fluid because as soon as a temperature difference is imposed in a fluid, natural convection currents will occur due to resulting density differences.

Convective heat transfer rate is governed by Newton's law of cooling and is expressed as

$$Q = h\,A_s\,(T_s - T_f) \qquad (3)$$

Where, 'h' is convective heat transfer coefficient in W/ (m$^2$ – K)

$A_s$ is heat transferring area, m$^2$

$T_s$ and $T_f$ are temperatures of surface and the fluid respectively, K

**Radiation:**

Heat exchanged between two bodies or mediums, which are separated and are not in contact with each other, is called radiation heat transfer. Radiation heat transfer does not require presence of an intervening medium between the two bodies as in case of conduction and convection and it takes place most effectively in a vacuum. Example of radiation heat transfer is the energy received on the earth from the Sun and has been shown in Figure 3.

Thermal radiation is the energy emitted by a body in the form of electromagnetic waves due to changes in the electronic configuration of the constituent atoms or molecules of the body. The electromagnetic waves travel through the intervening medium between two bodies with a speed that is related to speed of light in vacuum by the following equation

$$c = c_o / n. \tag{4}$$

Where c is the speed of propagation in a medium,

$C_o$ is the speed of light in vacuum and is equal to $3 \times 10^8$ m/sec,

'n' is an index of refraction of a medium which is unity for air and most of the gases, 1.5 for glass and 1.33 for water.

When electromagnetic waves come in contact with a body, energy is transferred to the body as thermal energy which is partly absorbed, reflected and transmitted.

Energy emitted per unit area as thermal radiation is called emissive power of a body and the maximum energy emitted as radiation by a body at a particular temperature is governed by Stefan-Boltzmann law which is expressed as

$$E_b = \sigma A T^4 \tag{5}$$

Where, $E_b$ is the energy emitted per unit time, W

A is the surface area, $m^2$

T is the absolute temperature of the body, K

σ is Stefan-Boltzmann constant which is equal to $5.67 \times 10^{-8}$ W/($m^2$ – $K^4$)

At a given temperature, maximum radiations are emitted by an ideal emitter called black body. The energy emitted by non-black bodies are less as compared to that of the ideal body when both the bodies are maintained at same temperature. Energy emitted by a non-black body maintained at temperature 'T' is given as

$$E = \varepsilon \sigma A T^4 \tag{6}$$

Where, ε is called emissivity of non-black bodyand is defined as ratio of emissive power of a non- black body to that of a black body. Emissivity is a radiative property of the body and its value depends upon surface characteristics and temperature of the body.

All the bodies radiate energy and receive energy emitted by other bodies simultaneously. Consider heat exchange between two black bodies maintained at temperatures $T_1$ and $T_2$ respectively and body 1 is completely enclosed by body 2. The net heat transfer by radiation from body 1 to body 2 is given as

$$Q_{1-2} = \sigma A_1 (T_1^4 - T_2^4) \tag{7}$$

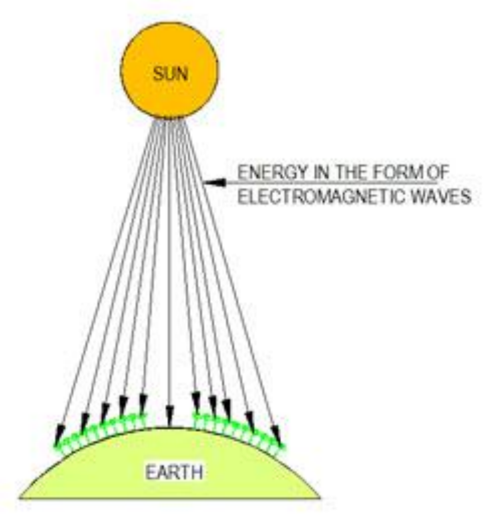



**Figure 3**

**Surface Energy Balance:**

Consider a cylinderical object of thickness 'L' which is being heated from inside and is surrounded by a fluid at temperature '$T_f$' and moving at a velocity 'V'. Heat is being transferred from inner surface to outer surface by conduction and conducted heat is transferred to surroundings by radiation and convection and has been depicted in Figure 4.

The energy balance equation for the arrangement can be expressed as

$$Q_{conducted} = Q_{convected} + Q_{Radiated}$$

$$-kA\frac{(T_1 - T_2)}{L} = hA(T_2 - T_f) + \varepsilon\sigma A\left(T_2^4 - T_{sur}^4\right) \tag{8}$$

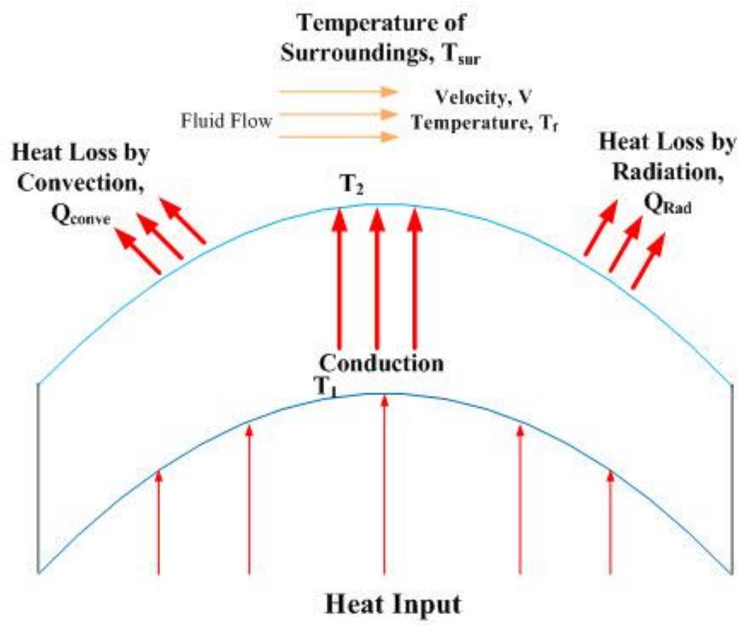



**Figure 4**

**Q. 1.1 Two surfaces of a plane wall of 15cm thickness and 5 m² area are maintained at 240°C and 90°C respectively. Determine the heat transfer between the surfaces and temperature gradient across the wall if conductivity of the wall material is18.5 W/(m-K).**

**Solution:** **Given : $T_1$= 2 40°C, $T_2$= 90°C, Thickness of wall, x = 15 cms.=0.15 m,**

**Area, A= 5 m², Thermal conducivity, k=18.5 W/(m-K)**

To determine: i) Temperature gradient, ,

ii) Heat transfer rate, ,

i) The temperature gradient in the direction of heat flow is

(b) Heat flow across the wall is given by Fourier's heat conduction equation

$$Q = -kA\frac{dt}{dx} = -18.5 \times 5 \times (-1000)$$

=92500 W or 92.50 kW

**Example 1.2 The bond between two plates, 2.5 cm and 15 cm thick, heat is uniformly applied through the thinner plate by a radiant heat source. The bonding epoxy must be held at 320 K for a short time. When the heat source is adjusted to have a steady value of 43.5 kW/m², a thermocouple installed on the side of the thinner plate next to source indicates a temperature of 345 K. Calculate the temperature gradient for heat conduction through thinner plate and thermal conductivity of its material.**

**Solution:** $t_1$ = 345 K ; $t_2$ = 320 K

δ = 2.5 cm = 0.025m

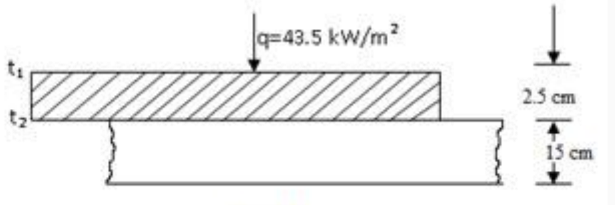



Fig. 1.4

Temperature gradient,

$$\frac{dt}{dx} = \frac{t_2 - t_1}{\delta} = \frac{320 - 345}{0.025} = -1000°C/m$$

**Example 1.3 A 120 W system is used to maintain a plate of 0.2m² area at a temperature of 60° C when the surroundings are at 30°C temperature. What fraction of heat supplied is lost by natural convection? It may be presumed that convection coefficient conforms to the relation h = 2.5 (ΔT)$^{0.2}$ W/m² K.**

**Solution:** Convective heat transfer coefficient, Heat lost by convection = h A Δt

= 4.9360.2 (60-30) = 29.616 W

Heat lost by convection as fraction of heat supplied,

$$=\frac{29.616}{120} = 0.2468 \; or \; 24.68\%$$

The remaining 75.32% would be lost to the surroundings by ration.

**Example 1.4 A cylindrical system,1 m long and 3 cm in diameter, is heated and positioned in a vacuum furnace which has interior walls at 900 K temperature.Current is passed through the rod and its surface is maintained at 1000K. Calculate the power supplied to the heating rod if its surface has an emissivity of 0.8.**

**Solution:** For steady state conditions, the electric power supplied to the rod equals the radiant heat loss from it. Further, since the walls of the furnace completely enclose the heating rod, all the radiant energy emitted by the surface of the rod is intercepted by the furnace walls. Thus

$$\begin{aligned} Q &= \sigma_b \, A\epsilon \, (T_1^4 - T_2^4) \\ &= \sigma_b \, (\pi dl) \, \epsilon \, (T_1^4 - T_2^4) \\ &= 5.67 \times 10^{-8} (\pi \times 0.03 \times 100)(0.8)(1100^4 - 900^4) \\ &= 5.67(\pi \times 0.03 \times 1.0)(0.8)\left[\left(\frac{1100}{100}\right)^4 - \left(\frac{900}{100}\right)^4\right] \\ &= 5.67(\pi \times 0.03 \times 1.0)(0.8)(14641 - 6561) = 3452.513W \end{aligned}$$

Thus the rate of electrical input to the rod must equal 3452.513 W.

**Example 1.5 A surface at 300°C loses heat both by convection and radiation to the surroundings at 150°C. The convection coefficient is 75 W/m²K and the radiation factor due to geometric location and emissivity is 0.85. If the heat is conducted to the surface through a solid material of thermal conductivity 15 W/mK, determine the temperature gradient at the surface of the solid.**

**Solution:** Under steady state conditions,

Heat convected + heat radiated = heat conducted

$$hA(T_1 - T_2) + FA\sigma_b(T_1^4 - T_2^4) = -kA\frac{dt}{dx}$$

Given: $T_1$=300+273=573 K and $T_2$=150+273=423 K

Considering unit area and substituting the appropriate values, we obtain

$$75 \times 1(573 - 423) + 0.85 \times 1 \times 5.67 \times 10^{-8}(573^4 - 423^4) = -15 \times 1 \times \frac{dt}{dx}$$

Or

11250+3652.43=

Thus, the temperature gradient at the surface of solid is

$$\frac{dt}{dx} = -\frac{11250+3652.43}{15} = -993.5K/m$$

## Lesson 2. Conduction- thermal conductivity of materials, General heat conduction equation

### Thermal Conductivity of Materials:

Thermal conductivity is basically an indicator of rate of flow of heat through a material. A higher value of thermal conductivity means that material is a good conductor of heat whereas a lower value means that material is a bad conductor of heat and will act as an insulator. Physical structure and density determine the value of thermal conductivity of a material. It is denoted by 'k' and is defined as rate of heat transfer through a unit thickness of the material per unit area and per unit temperature difference.

$$k = -\frac{Q_x}{A\left(\frac{dT}{dx}\right)} \quad (1)$$

**Solids:** Solids generally consist of free electrons and atoms bound in periodic arrangement called lattice. Thermal conductivity in solids is attributed to migration of free electrons and lattice vibrations. Therefore, thermal conductivity of a solid is the sum of electronic component and lattice vibration component. Thermal conductivity of pure metals is high and contribution of electronic component is significant as compared to that of lattice vibration component as shown in Figure 1. Thermal conductivity of an alloy of two metals is considerably lower that that of either of two metals. For example values of thermal conductivity of copper and Nickel are 401 W/(m- $^oK$) and 91 W/(m- $^oK$) respectively where as thermal conductivity of constantan ( 55% copper and 45% Nickel) is only 23 W/(m- $^oK$).

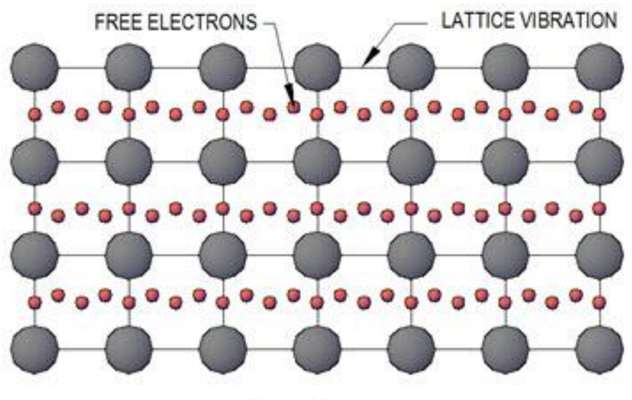



**Figure 1**

Arrangement of molecules in a material strongly affects the lattice component of thermal conductivity. Crystalline solid such as diamond has the highest value of thermal conductivity at room temperature. Thermal conductivity of solids generally decreases with increase in temperature.

**Fluids:** Solids generally have the highest value of thermal conductivity followed by liquids and gases on account of intermolecular spacing between the molecules. In solids molecules are closely packed, in liquids loosely packer and in gases very loosely packed. Thermal conductivity of gases increases with increase in temperature while that of liquids decreases with increase in temperature with glycerin and water being exceptions. Table 3 gives the values of thermal conductivity of different materials.

Table 3: Thermal Conductivity of Materials at Room Temperature

| S. No. | Material | Conductivity, k ,W/(m-K) |
|---|---|---|
| 1. | Diamond | 2300 |
| 2. | Silver | 429 |
| 3. | Copper | 401 |
| 4. | Gold | 317 |
| 5. | Aluminum | 237 |
| 6. | Iron | 80.2 |
| 7. | Glass | 0.78 |
| 8. | Water | 0.607 |
| 9. | Wood | 0.17 |
| 10. | Air | 0.026 |

**Objectives of conduction analysis:**

The basic objective of conduction analysis is to determine the variation of temperature with respect to location and time throughout a body. The knowledge of temperature distribution with in a body is required to determine heat transfer rate. Location of a point with in a body, at which temperature is to be determined, is specified by choosing a suitable coordinate system amongst Cartesian coordinates, cylindrical coordinates or spherical coordinates. Temperature at a point is expressed as

- $T(x, y, z, t)$ in rectangular or Cartesian coordinates for parallelepiped bodies
- $T(r, \varphi, z, t)$ in cylindrical coordinates for cylindrical bodies
- $T(r, \varphi, \theta, t)$ in spherical coordinates for spherical bodies.

For any irregular shaped body, generally rectangular or Cartesian coordinates are used for conduction analysis.

**(a) General Heat Conduction Equation in Cartesian Coordinates:**

Consider flow of heat through a very small control volume oriented into three dimensional. The sides dx, dy and dz are parallel to x, y and z axes respectively as shown in Figure 2.

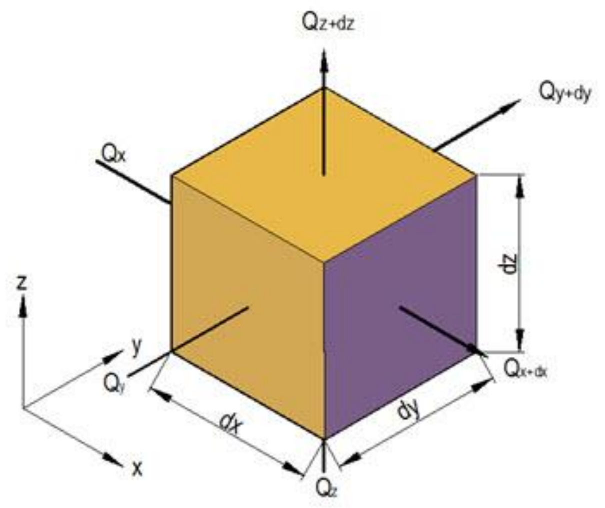



**Figure 2**

Let 'T' be the temperature of left face AEHD and it changes along x-direction. Let the temperature changes by amount ∂T through a unit distance ∂x and temperature gradient is expressed as $\frac{\partial T}{\partial x}$. Similarly, change of temperature through distance dx will be $\frac{\partial T}{\partial x}dx$ and temperature at the right face BFGC is $(T + \frac{\partial T}{\partial x}dx)$. For non-isotropic materials, thermal conductivity changes as heat flows through the control volume. If $K_x$ is the thermal conductivity at the left face AEHD, then quantity of heat flowing in to the control volume along x-direction through left face during time interval dt is expressed as

$$Q_x = -k_x A \frac{\partial T}{\partial x} dt = -k_x dydz \frac{\partial T}{\partial x} dt \qquad (2)$$

Where, A is the area of the left face ADHD

During same interval of time, heat flowing out of the right face BFGC along x-direction will be

$$Q_{x+dx} = Q_x + \frac{\partial (Q_x)}{\partial x} dx \qquad (3)$$

Heat accumulated in the control volume due to flow of heat along x-direction will be

$$q_x = Q_x - Q_{x+dx} = Q_x - \left( Q_x + \frac{\partial (Q_x)}{\partial x} dx \right)$$

$$= -\left( \frac{\partial (Q_x)}{\partial x} dx \right) = -\frac{\partial}{\partial x}\left( -k_x dy dz \frac{\partial T}{\partial x} dt \right) dx \tag{4}$$

$$= \frac{\partial}{\partial x}\left( k_x \frac{\partial T}{\partial x} \right) dx\, dy\, dz\, dt \tag{5}$$

Similarly, heat accumulated in the control volume due to heat flow along y and z directions can be expressed as

$$q_y = \frac{\partial}{\partial y}\left( k_y \frac{\partial T}{\partial y} \right) dx\, dy\, dz\, dt \tag{6}$$

$$q_z = \frac{\partial}{\partial z}\left( k_z \frac{\partial T}{\partial z} \right) dx\, dy\, dz\, dt \tag{7}$$

Let there is a heat source present inside the control volume which is generating '$q_g$' amount of heat per unit volume per unit time. Total heat generated in the control volume can be expressed as

$$= q_g dx\, dy\, dz\, dt \tag{8}$$

Total heat accumulate in the control volume is the sum of equations (5), (6), (7) and (8) and it results in increase in the thermal energy of the control volume. This increase in thermal energy is reflected by rate of change of heat capacity of the control volume during given time interval 'dt' and is expressed as

= Mass of the control volume X specific heat X change of temperature of control

volume per unit time X Time interval

$$= \rho (dx\, dy\, dz) C \frac{\partial T}{\partial t} dt \tag{9}$$

Heat Capacity or thermal capacity is defined as heat required to raise temperature of whole mass of a body by 1°C and is expressed as

= mass of a body X specific heat

From energy balance considerations, sum of equations (5), (6), (7) and (8) should be equal to equation (9).

$$(dx\,dy\,dz\,dt)\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)+q_g\right]=\rho(dx\,dy\,dz)C\frac{\partial T}{\partial t}dt \qquad (10)$$

$$\left[\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)+q_g\right]=\rho\, C\frac{\partial T}{\partial t} \qquad (11)$$

Equation (11) represents general conduction equation for three dimensional, unsteady heat flow through a non-isotropic material.

For homogeneous and isotropic materials, $k_x = k_y = k_z = k$. Therefore, Equation (11) will become

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_g}{k}=\frac{\rho C}{k}\frac{\partial T}{\partial t} \qquad (12)$$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{q_g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (13)$$

Where, $\alpha = \frac{k}{\rho C}$ and is called thermal diffusivity of a material and it represents how quickly heat will propagate through a material. Thermal diffusivity is defined as ratio of heat conducted by a material to the heat stored by it per unit volume.

$$\alpha = \frac{\text{Heat Conduted}}{\text{Heat Stored}}\ ,\ \text{m}^2/\text{sec}$$

Heat will propagate more quickly in a material having higher value of thermal diffusivity and materials with smaller thermal diffusivity will store most of the heat.

Equation (13) is known as **Fourier-Biot equation** and can be written as

$$\Delta^2 T+\frac{q_g}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (14)$$

$\Delta^2$ is called Laplacian operator.

Different Cases:

i) If flow of heat is under unsteady state conditions and no heat generation within the control volume takes place, then $q_g = 0$ and Equation (13) can be written as

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (15)$$

Equation (15) is known as **diffusion equation**

ii) If flow of heat is under steady state conditions, then temperature will not change with time and. Equation (13) can be written as

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0 \qquad (17)$$

Equation (16) is known as **Poisson equation**

iii) If flow of heat is under steady state conditions without heat generation, Equation (13) can be written as

$$\frac{\partial^2 T}{\partial x^2}=0 \qquad (18)$$

Equation (17) is known as **Laplace equation**

iv) For steady state and one-dimensional flow of heat through a body in which there is no heat generation, then Equation (13) can be written as

$$\frac{\partial^2 T}{\partial x^2}=0 \qquad (18)$$

**(b) General Heat Conduction Equation in Cylindrical Coordinates:**

When conductive heat transfer occurs through bodies having cylindrical geometries such as rods, pipes etc., general conduction equation is derived in cylindrical coordinates. Let us consider a control volume having cylindrical dimensions (r, φ, z) and dr, dz and rd are the three sides of the control volume as shown in Figure 3.

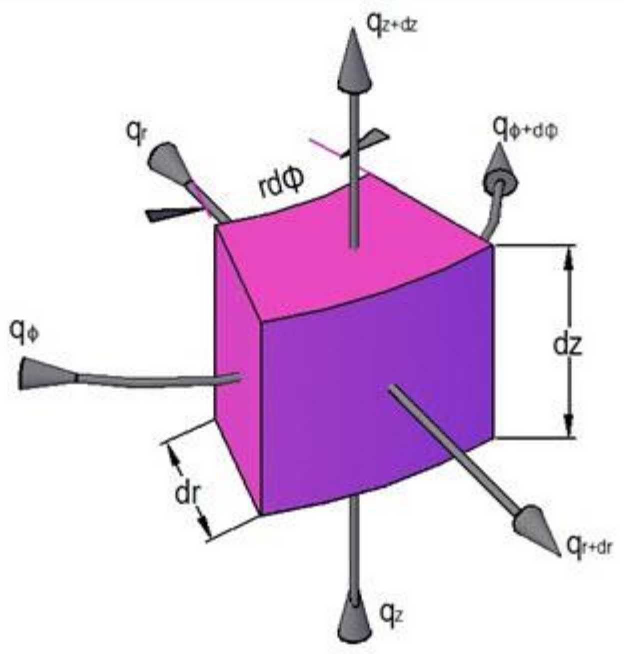



**Figure 3**

Heat inflow through (r, z) plane along direction is expressed as

$$Q_\phi = -k\,drdz\frac{\partial T}{r\partial\phi}dt \tag{19}$$

Heat outflow through (r, z) plane along direction is expressed as

$$Q_{\phi+d\phi} = Q_\phi + \frac{\partial(Q_\phi)}{r\partial\phi}rd\phi = -k\,drdz\frac{\partial T}{r\partial\phi}dt + \frac{\partial\left(-k\,drdz\frac{\partial T}{r\partial\phi}dt\right)}{r\partial\phi}rd\phi \tag{20}$$

Heat accumulated in control volume due to flow through (r-z) plane during time interval dt is expressed as

$$q_\phi = Q_\phi - Q_{\phi+d\phi} = -kdrdzd\phi\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial T}{\partial\phi}\right)dt \tag{21}$$

Heat inflow through (r,) plane along z-direction is expressed as

$$Q_z = -k\,rd\phi dr\frac{\partial T}{\partial z}dt \tag{22}$$

Heat outflow through (r,) plane along z-direction is expressed as

$$Q_{z+dz} = Q_z + \frac{\partial(Q_z)}{\partial z}dz = -k\,drrd\phi\frac{\partial T}{\partial z}dt + \frac{\partial\left(-k\,drrd\phi\frac{\partial T}{\partial z}dt\right)}{\partial z}dz \tag{23}$$

Heat accumulated in control volume due to flow through (r-) plane during time interval ‘dt’ is expressed as

$$q_z = Q_z - Q_{z+dz} = -krd\phi dr\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)dt \tag{24}$$

Heat inflow through (z- ) plane along r-direction is expressed as

$$Q_r = -k\,dzrd\phi\frac{\partial T}{\partial r}dt \tag{25}$$

Heat outflow through (z-) plane along r-direction is expressed as

$$Q_{r+dr} = Q_r + \frac{\partial(Q_r)}{\partial r}dr = -k\,rd\phi dz\frac{\partial T}{\partial r}dt + \frac{\partial\left(-k\,rd\phi dz\frac{\partial T}{\partial r}dt\right)}{\partial r}dr \tag{26}$$

Heat accumulated in control volume due to flow through (z-) plane during time interval dt is expressed as

$$q_r = Q_r - Q_{r+dr} = -k\,d\phi\,dr\,\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)dt \tag{27}$$

Let there is a heat source present inside the control volume which is generating $q_g$ amount of heat per unit volume per unit time. Total heat generated in the control volume can be expressed as

$$= q_g r\,d\phi\,dr\,dz\,dt \tag{28}$$

Total heat accumulate in the control volume is the sum of equations (21), (24), (27) and (28) and it results in increase in the thermal energy of the control volume. This increase in thermal energy is reflected by rate of change of heat capacity of the control volume during given time interval 'dt' and is expressed as

$$= \rho\,(r\,d\phi\,dr\,dz)\,C\frac{\partial T}{\partial t}dt \tag{29}$$

From energy balance considerations, sum of equations (21), (24), (27) and (28) should be equal to equation (29).

$$(k\,d\phi\,dr\,dz\,dt)\left[\frac{1}{r}\left(\frac{\partial^2 T}{\partial \phi^2}\right)+\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+r\frac{\partial^2 T}{\partial z^2}+\frac{q_g r}{k}\right]=\rho(r\,d\phi\,dr\,dz)C\frac{\partial T}{\partial t}dt \tag{30}$$

$$\left[\frac{1}{r}\left(\frac{\partial^2 T}{\partial \phi^2}\right)+r\frac{\partial^2 T}{\partial r^2}+\frac{\partial T}{\partial r}+r\frac{\partial^2 T}{\partial z^2}+\frac{q_g r}{k}\right]=\frac{\rho C}{k}r\,\frac{\partial T}{\partial t}dt \tag{31}$$

Replacing $\frac{\rho C}{k} = \alpha$ in Equation (31)

$$\left[\frac{1}{r^2}\left(\frac{\partial^2 T}{\partial \phi^2}\right)+\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}+\frac{q_g}{k}\right]=\frac{1}{\alpha}\,\frac{\partial T}{\partial t}dt \tag{32}$$

Equation (32) represents general conduction equation for three dimensional, unsteady heat flow in cylindrical coordinates.

**(C) General Heat Conduction Equation in Spherical Coordinates:**

Similarly, the general conduction equation for geometries having spherical coordinates can be obtained and is expressed as:

$$\left[\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 T}{\partial \phi^2}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{q_g}{k}\right]=\frac{1}{\alpha}\,\frac{\partial T}{\partial t}dt \tag{33}$$

**Lesson 3. One dimensional steady state conduction through plane and composite walls, tubes and spheres without heat generation**

**One-Dimensional, Steady State Heat Conduction without Heat Generation:**

**i) Plane Wall or Slab of Uniform Conductivity without Heat Generation:**

Consider steady state heat conduction through a plane wall of thickness 'L' and area 'A' having uniform conductivity 'k' as shown in Figure 1. Temperature on the left hand side of the wall is $T_1$ and on the right hand side it is $T_2$. Heat is flowing from left hand side to the right hand side as $T_1$ is greater than $T_2$. The general conduction equation which governs the conduction heat transfer is written as

$$\left[\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z \frac{\partial T}{\partial z}\right)+q_g\right]=\rho\, C\frac{\partial T}{\partial t} \qquad (1)$$

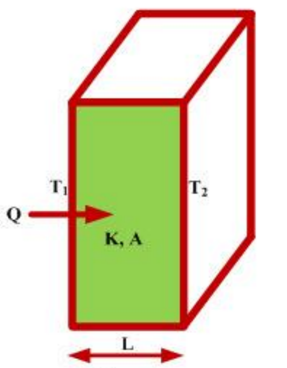


Figure 1

Since it is a case of one-dimensional, stead heat conduction through a wall of uniform conductivity without heat generation, therefore, $\frac{dT}{dt}=0$ , $\frac{dT}{dy}=\frac{dT}{dz}=0$ and $q_g=0$ and

Therefore, equation (1) reduces to

$$\frac{d^2T}{dx^2}=0 \qquad (2)$$

Equation (2) is used to determine the temperature distribution and heat transfer rate through the wall. Integrating equation (2) twice with respect to x, it can written as

$$T = C_1\, x + C_2 \qquad (3)$$

Where, $C_1$ and $C_2$ are constants of integration.

Using the following boundary conditions:

i. At $x = 0$, $T = T_1$

Equation (3) is written as $C_2 = T_1$ (4)

ii. At $x = L$, $T = T_2$

Equation (3) can be written as $T_2 = C_1 L + C_2$

Or $T_2 = C_1 L + T_1$

$C_1 = (T_2 - T_1)/L$ (5)

Substituting the values of $C_1$ and $C_2$ in equation (3)

$$T = \frac{T_2 - T_1}{L} x + T_1$$

Or
$$T = T_1 - \frac{T_1 - T_2}{L} x \qquad (6)$$

Equation (6) represents temperature distribution in the wall. It means temperature at any point along the thickness of the wall can be obtained if values of temperatures $T_1$, $T_2$, thickness L and distance of the point form either of the faces of the wall are known.

Rate of heat transfer can be determined by using Fourier's law and can be expressed as

$$Q = -kA\frac{dT}{dx} \qquad (7)$$

Integrating equation (6) with respect to x to obtain the expression for temperature gradient $\frac{dT}{dx}$

$$\frac{d}{dx}\int T = \frac{d}{dx}\int\left(T_1 - \frac{T_1 - T_2}{L}x\right)$$
$$\frac{dT}{dx} = -\frac{T_1 - T_2}{L}x$$
$$\frac{dT}{dx} = \frac{T_2 - T_1}{L}$$

Substituting the value of $\frac{dT}{dx}$ from above equation in equation (7), we get

$$Q = -kA\left(\frac{T_2 - T_1}{L}\right) \qquad (8)$$

Equation (8) represents the heat transfer rate through the wall.

**ii) Cylinder of Uniform Conductivity without Heat Generation:**

Consider steady state heat conduction through a cylinder having $r_1$ and $r_2$ as inner and outer radii respectively and length 'L' as shown in Figure 2. Temperature of the inner and outer surfaces is $T_1$ and $T_2$ respectively. Heat is flowing from inner to outer surface as $T_1$ is greater than $T_2$. The general conduction equation which governs the conduction heat transfer is written as

$$\left[\frac{1}{r^2}\left(\frac{\partial^2 T}{\partial \phi^2}\right)+\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}+\frac{q_g}{k}\right]=\frac{1}{\alpha}\;\frac{\partial T}{\partial t}dt \qquad (9)$$

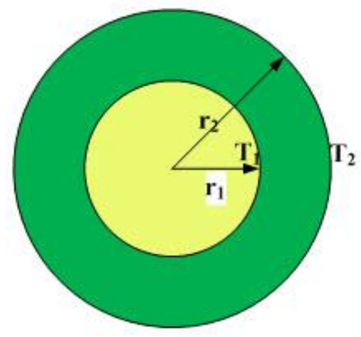



Figure 2

Since it is a case of one-dimensional, stead heat conduction through a wall of uniform conductivity without heat generation, therefore, $\frac{dT}{dt}=0$ , $\frac{dT}{d\phi}=\frac{dT}{dz}=0$ and $q_g=0$

Therefore, equation (9) reduces to

$$\frac{d^2T}{dr^2}+\frac{dT}{dr}=0$$

Or

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0 \qquad (10)$$

Equation (10) is used to determine the temperature distribution and heat transfer rate through the cylinder. Integrating equation (10) twice with respect to r, it can written as

$$r\frac{dT}{dr}=C_1 \text{ or } \frac{dT}{dr}=\frac{C_1}{r} \qquad (11)$$

and $T = C_1 \log_e r + C_2$ (12)

Using the following boundary conditions:

i. At $r = r_1$, $T = T_1$

Equation (12) is written as $T_1 = C_1 \log_e r_1 + C_2$ (13)

ii. At $r = r_2$, $T = T_2$

Equation (12) can be written as

$T_2 = C_1 \log_e r_2 + C_2$ (14)

Subtracting equation (14) from equation (13), we get

$T_1 - T_2 = C_1 \log_e r_1 - C_1 \log_e r_2$

$$T_1 - T_2 = C_1 \log_e \frac{r_1}{r_2}$$

$$C_1 = \frac{T_1 - T_2}{\log_e \frac{r_1}{r_2}} \qquad (15)$$

Substituting the values of $C_1$ from equation (15) in equation (13)

$$T_1 = \frac{T_1 - T_2}{\log_e \frac{r_1}{r_2}} \log_e r_1 + C_2 \qquad (16)$$

$$C_2 = T_1 - \frac{T_1 - T_2}{\log_e \frac{r_1}{r_2}} \log_e r_1$$

$$= \frac{T_1 \log_e r_1 - T_1 \log_e r_2 - T_1 \log_e r_1 + T_2 \log_e r_1}{\log_e \frac{r_1}{r_2}}$$

$$= \frac{T_2 \log_e r_1 - T_1 \log_e r_2}{\log_e \frac{r_1}{r_2}}$$

$$C_2 = \frac{T_1 \log_e r_2 - T_2 \log_e r_1}{\log_e \frac{r_2}{r_1}} \qquad (17)$$

Substituting the values of $C_1$ and $C_2$ in equation (12), we get

$$T = \frac{T_1 - T_2}{\log_e \frac{r_1}{r_2}} \log_e r + \frac{T_1 \log_e r_2 - T_2 \log_e r_1}{\log_e \frac{r_2}{r_1}}$$

$$T = \frac{1}{\log_e \frac{r_2}{r_1}} \left[ T_1 \log_e r_2 - T_2 \log_e r_1 - (T_1 - T_2) \log_e r \right]$$

$$T = \frac{1}{\log_e \frac{r_2}{r_1}} \left[ T_1 \log_e \frac{r_2}{r} - T_2 \log_e \frac{r}{r_1} \right] \tag{18}$$

Equation (18) represents temperature distribution in the cylinder. Rate of heat transfer can be determined by using Fourier's law and can be expressed as

$$Q = \left[ -kA \frac{dT}{dr} \right]_{r=r_1} \tag{19}$$

$$Q = -k \times 2\pi r_1 \times L_1 \left( \frac{dT}{dr} \right)_{r=r_1} \tag{20}$$

From equation (11) we can write

Substituting the value of $C_1$ from equation (15), we can write

$$\frac{dT}{dr} = \frac{T_1 - T_2}{\log_e \frac{r_1}{r_2}} \times \frac{1}{r}$$

At $r = r_1$,

$$\frac{dT}{dr} = \frac{T_1 - T_2}{\log_e \frac{r_1}{r_2}} \times \frac{1}{r_1} \tag{21}$$

Substituting the value of from equation (21) in equation (20), we get

$$Q = -k \times 2\pi r_1 \times L_1 \left( \frac{T_1 - T_2}{\log_e \frac{r_1}{r_2}} \times \frac{1}{r_1} \right)$$

$$Q = -k \times 2\pi \times L_1 \left( \frac{T_1 - T_2}{\log_e \frac{r_1}{r_2}} \right)$$

$$Q = \frac{2\pi k \times L_1 (T_2 - T_1)}{\log_e \frac{r_2}{r_1}} \qquad (22)$$

Equation (22) represents the heat transfer rate through the cylinder.

**iii) Sphere of Uniform Conductivity without Heat Generation:**

Consider steady state heat conduction through a hollow sphere having $r_1$ and $r_2$ as inner and outer radii respectively. Temperature of the inner and outer surfaces is $T_1$ and $T_2$ respectively. Heat is flowing from inner to outer surface as $T_1$ is greater than $T_2$. The general conduction equation which governs the conduction heat transfer is written as

$$\left[ \frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial^2 T}{\partial \phi^2}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{q_g}{k} \right] = \frac{1}{\alpha}\frac{\partial T}{\partial t} dt \qquad (22)$$

Since it is a case of one-dimensional, steady heat conduction through a shere without heat generation, therefore, $\frac{dT}{dt} = 0$ , $\frac{dT}{d\phi} = \frac{dT}{d\theta} = 0$ and $q_g = 0$

Therefore, equation (22) reduces to

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} = 0 \qquad (23)$$

Multiplying both sides of equation (23) by $r^2$ , we get

$$r^2 \frac{d^2T}{dr^2} + 2r\frac{dT}{dr} = 0$$

Or
$$\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0 \qquad (24)$$

Equation (24) is used to determine the temperature distribution and heat transfer rate through the wall. Integrating equation (23) twice with respect to r, it can written as

$$r^2\frac{dT}{dr}=c_1 \text{ or } \frac{dT}{dr}=\frac{c_1}{r^2} \qquad (25)$$

and $$T=-\frac{C_1}{r}+C_2 \qquad (26)$$

Using the following boundary conditions:

i. At r = $r_1$, T = $T_1$

Equation (26) is written as

$$T_1=-\frac{C_1}{r_1}+C_2 \qquad (27)$$

ii. At r = $r_2$, T = $T_2$

Equation (26) can be written as

$$T_2=-\frac{C_1}{r_2}+C_2 \qquad (28)$$

Subtracting equation (28) from equation (27), we get

$$T_1-T_2=C_1\left(\frac{1}{r_2}-\frac{1}{r_1}\right)$$

$$C_1=\frac{T_1-T_2}{\left(\frac{1}{r_2}-\frac{1}{r_1}\right)} \qquad (29)$$

Substituting the values of $C_1$ from equation (29) in equation (27)

$$C_2=T_1+\frac{1}{r_1}\frac{T_1-T_2}{\left(\frac{1}{r_2}-\frac{1}{r_1}\right)} \qquad (30)$$

Substituting the values of $C_1$ and $C_2$ from equations (29) and (30) in equation (26)

$$T = -\frac{1}{r}\frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} + T_1 + \frac{1}{r_1}\frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$$

$$T = T_1 - \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}\left(\frac{1}{r} - \frac{1}{r_1}\right) \qquad (31)$$

Equation (31) represents temperature distribution in a sphere. Rate of heat transfer can be determined by using Fourier's law and can be expressed as

$$Q = \left[-kA\frac{dT}{dr}\right]_{r=r_1} \qquad (32)$$

$$Q = -k \times 4\pi r^2\left(\frac{dT}{dr}\right)_{r=r_1} \qquad (33)$$

From equation (25) we can write

Substituting the value of $C_1$ from equation (29), we can write $\frac{dT}{dr} = \frac{c_1}{r^2}$

$$\frac{dT}{dr} = \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \times \frac{1}{r^2}$$

At $r = r_1$

$$\frac{dT}{dr} = \frac{T_1 - T_2}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)} \times \frac{1}{r_1^2} \qquad (34)$$

Substituting the value of $\frac{dT}{dr}$ from equation (34) from equation (33), we get

$$Q = -k \times 4\pi r^2 \times \frac{T_1 - T_2}{r^2}\frac{1}{\left(\frac{1}{r_2} - \frac{1}{r_1}\right)}$$

$$Q = \frac{4\pi r_1 r_2 k(T_1 - T_2)}{(r_2 - r_1)} \qquad (35)$$

Equation (35) represents the heat transfer rate through a sphere.

**Heat Flow through Composite Geometries:**

**A) Composite Slab or Wall:**

Consider a composite slab made of three different materials having conductivity $k_1$, $k_2$ and $k_3$, length $L_1$, $L_2$ and $L_3$ as shown in Figure 3. One side of the wall is exposed to a hot fluid having temperature $T_f$ and on the other side is atmospheric air at temperature $T_a$. Convective heat transfer coefficient between the hot fluid and inside surface of wall is $h_i$ (inside convective heat transfer coefficient) and $h_o$ is the convective heat transfer coefficient between atmospheric air and outside surface of the wall (outside convective heat transfer coefficient). Temperatures at inner and outer surfaces of the composite wall are $T_1$ and $T_4$ whereas at the interface of the constituent materials of the slab are $T_2$ and $T_3$ respectively.

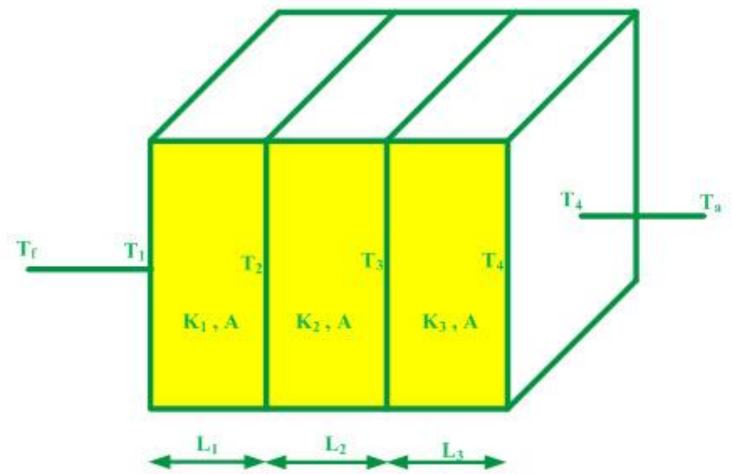



Figure 3

Heat is transferred from hot fluid to atmospheric air and involves following steps:

**i) Heat transfer from hot fluid to inside surface of the composite wall by convection**

$$Q = h_i\, A\left(T_f - T_1\right)$$

$$\frac{Q}{h_i\, A} = \left(T_f - T_1\right) \qquad (36)$$

**ii) Heat transfer from inside surface to first interface by conduction**

$$Q = \frac{k_1 A\left(T_1 - T_2\right)}{L_1}$$

$$\frac{Q}{\frac{k_1 A}{L_1}} = \left(T_1 - T_2\right) \qquad (37)$$

**iii) Heat transfer from first interface to second interface by conduction**

$$Q = \frac{k_2 A (T_2 - T_3)}{L_2}$$

$$\frac{Q}{\frac{k_2 A}{L_2}} = (T_2 - T_3) \qquad (38)$$

**iv) Heat transfer from second interface to outer surface of the composite wall by conduction**

$$Q = \frac{k_3 A (T_3 - T_4)}{L_3}$$

$$\frac{Q}{\frac{k_3 A}{L_3}} = (T_3 - T_4) \qquad (39)$$

**v) Heat transfer from outer surface of composite wall to atmospheric air by convection**

$$Q = h_o A (T_4 - T_a)$$

$$\frac{Q}{h_o A} = (T_4 - T_a) \qquad (40)$$

Adding equations (36), (37), (38) and (40), we get

$$Q\left(\frac{1}{h_i A} + \frac{1}{\frac{k_1 A}{L_1}} + \frac{1}{\frac{k_2 A}{L_2}} + \frac{1}{\frac{k_3 A}{L_3}} + \frac{1}{h_o A}\right) = (T_f - T_1 + T_1 - T_2 + T_2 - T_3 + T_3 - T_4 + T_4 - T_a)$$

or

$$Q\left(\frac{1}{h_i A} + \frac{1}{\frac{k_1 A}{L_1}} + \frac{1}{\frac{k_2 A}{L_2}} + \frac{1}{\frac{k_3 A}{L_3}} + \frac{1}{h_o A}\right) = (T_f - T_a)$$

or

$$Q = \frac{(T_f - T_a)}{\frac{1}{h_i A} + \frac{1}{\frac{k_1 A}{L_1}} + \frac{1}{\frac{k_2 A}{L_2}} + \frac{1}{\frac{k_3 A}{L_3}} + \frac{1}{h_o A}} \qquad (41)$$

If composite slab is made of 'n' number of materials, then equation (41) reduces to

$$Q = \frac{(T_f - T_a)}{\frac{1}{A}\left(\frac{1}{h_i} + \frac{1}{h_o} + \sum_{n=1}^{n=n}\left(\frac{L_n}{k_n}\right)\right)} \qquad (42)$$

If inside and outside convective heat transfer coefficients are not to be considered, then equation (42) is expressed as

$$Q = \frac{\left(T_f - T_a\right)}{\frac{1}{A}\left(\sum_{n=1}^{n=n}\left(\frac{L_n}{k_n}\right)\right)} \qquad (43)$$

**B) Composite Cylinder:**

Consider a composite cylinder consisting of inner and outer cylinders of radii $r_1$, $r_2$ and thermal conductivity $k_1$, $k_2$ respectively as shown in Figure 4. Length of the composite cylinder is L. Hot fluid at temperature $T_f$ is flowing inside the composite cylinder. Temperature at the inner surface of the composite cylinder exposed to hot fluid is $T_1$ and outer surface of the composite cylinder is at temperature $T_3$ and is exposed to atmospheric air at temperature $T_a$. The interface temperature of the composite cylinder is $T_2$.. Convective heat transfer coefficient between the hot fluid and inside surface of composite cylinder is $h_i$ (inside convective heat transfer coefficient) and $h_o$ is the convective heat transfer coefficient between atmospheric air and outside surface of the composite cylinder (outside convective heat transfer coefficient). Heat is transferred from hot fluid to atmospheric air and involves following steps:

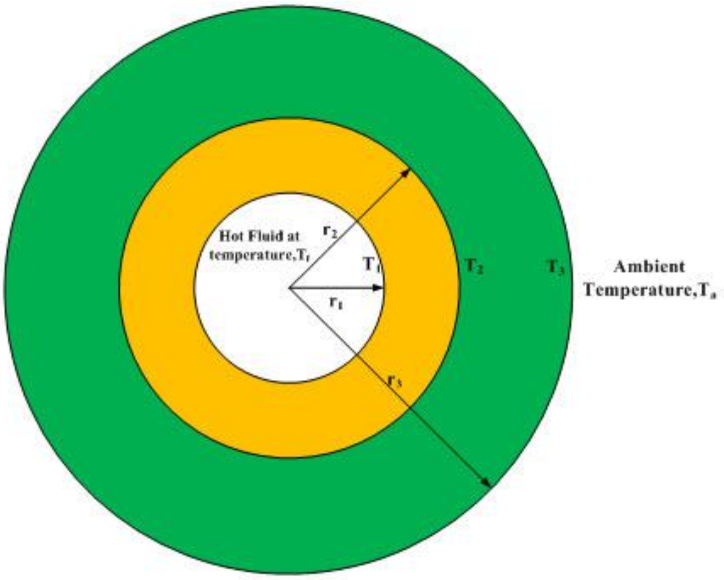



Figure 4

**i) Heat transfer from hot fluid to inside surface of the composite cylinder by convection**

$$Q = h_i\, A\left(T_f - T_1\right)$$

$$Q = h_i\, 2\pi r_1 L\left(T_f - T_1\right)$$

$$\frac{Q}{h_i\, 2\pi r_1 L} = \left(T_f - T_1\right) \qquad (44)$$

**ii) Heat transfer from inside surface to interface by conduction**

$$Q = \frac{k_1 2\pi L (T_1 - T_2)}{\log_e \frac{r_2}{r_1}}$$

$$\frac{Q}{\frac{k_1 2\pi L}{\log_e \frac{r_2}{r_1}}} = (T_1 - T_2) \qquad (45)$$

**iii) Heat transfer from interface to outer surface of the composite cylinder by conduction**

$$Q = \frac{k_2 2\pi L (T_2 - T_3)}{\log_e \frac{r_3}{r_2}}$$

$$\frac{Q}{\frac{k_2 2\pi L}{\log_e \frac{r_3}{r_2}}} = (T_2 - T_3) \qquad (46)$$

**iv) Heat transfer from outer surface of composite wall to atmospheric air by convection**

$$Q = h_o\, 2\pi r_3 L (T_4 - T_a)$$

$$\frac{Q}{h_o\, 2\pi r_3 L} = (T_4 - T_a) \qquad (47)$$

Adding both sides of equations (44), (45),(46) and (47), we get

$$\frac{Q}{2\pi L}\left(\frac{1}{h_i r_1}+\frac{1}{\dfrac{k_1}{Log_e \dfrac{r_2}{r_1}}}+\frac{1}{\dfrac{k_2}{Log_e \dfrac{r_3}{r_2}}}+\frac{1}{h_o r_3}\right)=\left(T_f-T_1+T_1-T_2+T_2-T_3+T_3-T_a\right)$$

or

$$Q\left(\frac{1}{h_i r_1}+\frac{1}{\dfrac{k_1}{Log_e \dfrac{r_2}{r_1}}}+\frac{1}{\dfrac{k_2}{Log_e \dfrac{r_3}{r_2}}}+\frac{1}{h_o r_3}\right)=2\pi L\left(T_f-T_a\right)$$

or

$$Q=\frac{2\pi L\left(T_f-T_a\right)}{\left(\dfrac{1}{h_i r_1}+\dfrac{1}{\dfrac{k_1}{Log_e \dfrac{r_2}{r_1}}}+\dfrac{1}{\dfrac{k_2}{Log_e \dfrac{r_3}{r_2}}}+\dfrac{1}{h_o r_3}\right)} \tag{48}$$

If the composite cylinder consists of 'n' cylinders, then equation (48) can be expressed as:

$$Q=\frac{2\pi L\left(T_f-T_a\right)}{\left(\dfrac{1}{h_i r_1}+\dfrac{1}{h_o r_{n+1}}+\sum_{n=1}^{n=n}\dfrac{1}{k_n}Log_e\left(\dfrac{r_{n+1}}{r_n}\right)\right)} \tag{49}$$

If inside and outside convective heat transfer coefficients are not to be considered, then equation (3.41) is expressed as

$$Q=\frac{2\pi L\left(T_f-T_a\right)}{\left(\sum_{n=1}^{n=n}\dfrac{1}{k_n}Log_e\left(\dfrac{r_{n+1}}{r_n}\right)\right)} \tag{50}$$

**C) Composite Sphere:**

Consider a composite sphere consisting of inner and outer cylinders of radii $r_1$, $r_2$ and thermal conductivity $k_1$, $k_2$ respectively. Hot fluid at temperature $T_f$ is flowing inside the composite sphere. Temperature at the inner surface of the composite sphere exposed to hot fluid if $T_1$ and outer surface of the composite cylinder is at temperature $T_3$ and is exposed to atmospheric air at temperature $T_a$. The interface temperature of the composite cylinder is $T_2$. Convective heat transfer coefficient between the hot fluid and inside surface of composite sphere is $h_i$ (inside convective heat transfer coefficient) and $h_o$ is the convective heat transfer coefficient between atmospheric air and outside surface of the composite sphere (outside convective heat transfer coefficient). Heat is transferred from hot fluid to atmospheric air and involves following steps:

**i) Heat transfer from hot fluid to inside surface of the composite sphere by convection**

$$Q = h_i A \left(T_f - T_1\right)$$

$$Q = h_i 4\pi r_1^2 \left(T_f - T_1\right)$$

$$\frac{Q}{h_i 4\pi r_1^2} = \left(T_f - T_1\right) \qquad (51)$$

**ii) Heat transfer from inside surface to interface by conduction.**

$$Q = \frac{4\pi k_1 r_1 r_2 \left(T_1 - T_2\right)}{r_2 - r_1}$$

$$\frac{Q\left(r_2 - r_1\right)}{4\pi k_1 r_1 r_2} = \left(T_1 - T_2\right) \qquad (52)$$

**iii) Heat transfer from interface to outer surface of the composite sphere by conduction**

$$Q = \frac{4\pi k_2 r_2 r_3 \left(T_2 - T_3\right)}{r_3 - r_2}$$

$$\frac{Q\left(r_3 - r_2\right)}{4\pi k_2 r_2 r_3} = \left(T_2 - T_3\right) \qquad (53)$$

**iv) Heat transfer from outer surface of composite wall to atmospheric air by convection**

$$Q = h_o 4\pi r_3^2 \left(T_4 - T_a\right)$$

$$\frac{Q}{h_o 4\pi r_3^2} = \left(T_4 - T_a\right) \qquad (54)$$

Adding both sides of equations (51), (52),(53) and (54), we get

$$\frac{Q}{4\pi}\left(\frac{1}{h_i r_1^2}+\frac{1}{\frac{(r_2-r_1)}{k_1 r_1 r_2}}+\frac{1}{\frac{(r_3-r_2)}{k_2 r_2 r_3}}+\frac{1}{h_o r_3^2}\right)=\left(T_f-T_1+T_1-T_2+T_2-T_3+T_3-T_a\right)$$

or

$$Q\left(\frac{1}{h_i r_1^2}+\frac{1}{\frac{(r_2-r_1)}{k_1 r_1 r_2}}+\frac{1}{\frac{(r_3-r_2)}{k_2 r_2 r_3}}+\frac{1}{h_o r_3^2}\right)=4\pi\left(T_f-T_a\right)$$

or

$$Q=\frac{4\pi\left(T_f-T_a\right)}{\left(\frac{1}{h_i r_1^2}+\frac{1}{\frac{(r_2-r_1)}{k_1 r_1 r_2}}+\frac{1}{\frac{(r_3-r_2)}{k_2 r_2 r_3}}+\frac{1}{h_o r_3^2}\right)} \qquad (54)$$

If the composite sphere consists of 'n' concentric spheres, then equation (54) can be expressed as:

$$Q=\frac{2\pi L\left(T_f-T_a\right)}{\left(\frac{1}{h_i r_1^2}+\frac{1}{h_o\left(r_{n+1}\right)^2}+\sum_{n=1}^{n=n}\left(\frac{r_{(n+1)}-r_n}{k_n r_n r_{(n+1)}}\right)\right)} \qquad (55)$$

If inside and outside convective heat transfer coefficients are not to be considered, then equation (55) is expressed as

$$Q=\frac{4\pi\left(T_1-T_{n+1}\right)}{\left(\sum_{n=1}^{n=n}\left(\frac{r_{(n+1)}-r_n}{k_n r_n r_{(n+1)}}\right)\right)}$$

**Lesson 4. One dimensional steady state conduction through plane and composite walls, tubes and spheres with heat generation**

**One-Dimensional, Steady State Heat Conduction with Heat Generation:**

One dimensional, steady state heat conduction is considered for following geometries

1) Slab

2) Cylinder

3) Sphere

**1) One-Dimensional Heat Flow through a slab with Heat Generation:**

**i) When Temperature of Both Sides of Slab is Same:**

Consider a slab of thickness 'L' and cross-sectional area 'A' through which heat flow takes place in x-direction. A heat source located at the center of the slab is generating '$q_g$' amount of heat per unit volume per unit time as shown in Figure 1.

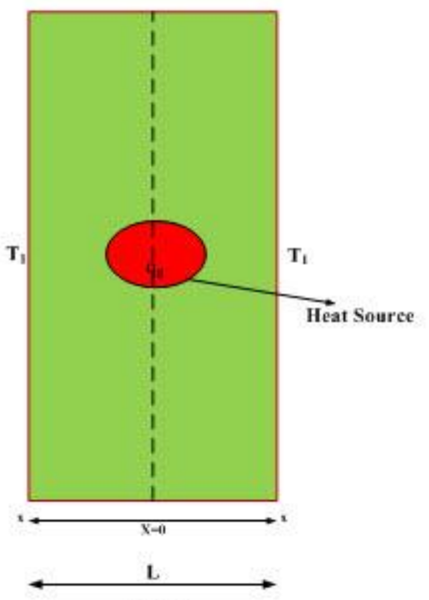


Figure 1

Heat generated is conducted equally towards the sides of the slab through a distance 'x' measured from center of the slab along x-direction. Temperature of both sides of the slab is same and is equal to $T_1$ as same amount of heat is flowing from the center towards the sides of the slab.

At the center of the slab x=0 and at the sides of the slab x= L/2. The general conduction equation under the given conditions reduces to

$$\frac{d^2T}{dx^2}+\frac{q_g}{k}=0 \qquad (1)$$

Integrating equation (1) with respect to 'x', we get

$$\frac{dT}{dx}+\frac{q_g}{k}x=C_1 \qquad (2)$$

Integrating equation (2) again with respect to 'x', we get

$$T+\frac{q_g}{k}\frac{x^2}{2}=C_1x+C_2 \qquad (3)$$

Using the boundary conditions,

At x = 0,  , From equation (2), we get

At $x = 0$, $\frac{dT}{dx}=0$, From equation (2), we get

$$C_1=0 \qquad (4)$$

At x = L/2, T = $T_1$, From equation (3), we get

$$T_1+\frac{q_g}{k}\frac{L^2}{8}=C_1\frac{L}{2}+C_2$$

As $C_1=0$, we can write

$$T_1+\frac{q_g}{k}\frac{L^2}{8}=C_2 \qquad (5)$$

Substituting the values of  and  in equation (3), we get

$$T+\frac{q_g}{k}\frac{x^2}{2}=T_1+\frac{q_g}{k}\frac{L^2}{8}$$

$$T=T_1+\frac{q_g}{k}\frac{L^2}{8}\left(1-\left(\frac{2x}{L}\right)^2\right) \qquad (6)$$

Equation (6) represents temperature distribution equation in the slab having a heat generating source present inside it.

Temperature will be maximum at the center of the slab where x = 0

$$T_{max} = T_1 + \frac{q_g}{k}\frac{L^2}{8} \tag{7}$$

Flow of heat can be expressed as:

$$Q = -kA\left(\frac{dT}{dx}\right)_{x=L/2} \tag{8}$$

Using equation (2), we can write

$$\left(\frac{dT}{dx}\right)_{x=L/2} = -\frac{q_g}{k}\frac{L}{2} \tag{9}$$

Substituting value of $\left(\frac{dT}{dx}\right)_{x=L/2}$ from equation (9) in equation (8), we get

$$Q = -kA\left(-\frac{q_g}{k}\frac{L}{2}\right)$$

$$Q = AL\frac{q_g}{2} \tag{10}$$

Equation (10) represents flow from one of sides of the slab; therefore, total heat flow from both the sides is expressed as

$$Q_{Total} = 2\times AL\frac{q_g}{2} = ALq_g$$

Total Heat Conducted form both sides of the slab = Volume x Heat generating capacity

**Total Heat Conducted from both sides of the slab = Total Heat generated**

Under steady state conditions, heat conducted at x = L/2 must be equal to convected from a side to the atmospheric air. Therefore,

$$Q = AL\frac{q_g}{2} = hA(T_1 - T_a)$$

$$T_1 = T_a + \frac{q_g L}{2h} \tag{11}$$

Substituting the value of $T_1$ from equation (11) in equation (6), we get

$$T = T_a + \frac{q_g L}{2h} + \frac{q_g}{k}\frac{L^2}{8}\left(1 - \left(\frac{2x}{L}\right)^2\right) \tag{12}$$

Equation (12) represents temperature distribution. If one side of the slab is insulated

At one side, temperature distribution will be represented by equation (6) except that L/2 will be replaced by L and is expressed as

$$T = T_1 + \frac{q_g}{k}\frac{L^2}{8}\left(1 - \left(\frac{2x}{2L}\right)^2\right)$$

$$T = T_1 + \frac{q_g}{k}\frac{L^2}{8}\left(1 - \left(\frac{x}{L}\right)^2\right)$$

**ii) When Temperature of Both Sides of Slab are Different:**

If the heat source present inside the slab generates heat $q_g$ per unit volume and heat distribution in towards both slabs is not uniform then the temperature of both sides of the slab will be different as shown in Figure 2.

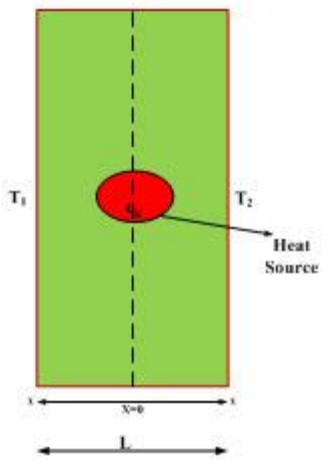



Figure 2

The differential equation governing the heat flow through the slab is expressed as:

$$\frac{d^2T}{dx^2} + \frac{q_g}{k} = 0 \qquad (13)$$

Integrating equation (13) with respect to 'x', we get

$$\frac{dT}{dx} + \frac{q_g}{k}x = C_1 \qquad (14)$$

Integrating equation (4.14) again with respect to 'x', we get

$$T + \frac{q_g}{k}\frac{x^2}{2} = C_1 x + C_2 \qquad (15)$$

Applying the boundary conditions,

At $x = 0$, $T = T_1$, From equation (15), we get

$$C_2 = T_1 \tag{16}$$

At $x = L$, $T = T_2$, From equation (15), we get

$$\frac{T_2 - T_1}{L} + \frac{q_g L}{2k} = C_1 \tag{17}$$

Substituting the values of and in equation (15), we get

$$T + \frac{q_g}{k}\frac{x^2}{2} = x\left[\frac{T_2 - T_1}{L} + \frac{q_g L}{2k}\right] + T_1$$

$$T = T_1 + \frac{q_g L x}{2k} - \frac{q_g x^2}{2k} + \frac{x}{L}(T_2 - T_1) \tag{18}$$

Subtracting $T_2$ from both sides of the equation (18), we get

$$T - T_2 = T_1 - T_2 + \frac{q_g L x}{2k} - \frac{q_g x^2}{2k} + \frac{x}{L}(T_2 - T_1)$$

$$T - T_2 = T_1 - T_2 + \frac{q_g}{2k}(Lx - x^2) + \frac{x}{L}(T_2 - T_1)$$

$$T - T_2 = T_1 - T_2 + \frac{q_g}{2k}(Lx - x^2) + \frac{x}{L}(T_2 - T_1) \tag{19}$$

Dividing both sides of the equation (19) with $T_1 - T_2$, we get

$$\frac{T-T_2}{T_1-T_2}=1+\frac{q_g}{2k(T_1-T_2)}(Lx-x^2)+\frac{x}{L}\frac{(T_2-T_1)}{(T_1-T_2)}$$

$$\frac{T-T_2}{T_1-T_2}=1+\frac{q_g}{2k(T_1-T_2)}(Lx-x^2)-\frac{x}{L}\frac{(T_1-T_2)}{(T_1-T_2)}$$

$$\frac{T-T_2}{T_1-T_2}=1-\frac{x}{L}+\frac{q_g}{2k(T_1-T_2)}(Lx-x^2)$$

$$\frac{T-T_2}{T_1-T_2}=1-\frac{x}{L}+\frac{q_g}{2k(T_1-T_2)}(Lx-x^2)\frac{L^2}{L^2}$$

$$\frac{T-T_2}{T_1-T_2}=1-\frac{x}{L}+\frac{q_g L^2}{2k(T_1-T_2)}\left[\frac{x}{L}-\left(\frac{x}{L}\right)^2\right]$$

$$\frac{T-T_2}{T_1-T_2}=\left(1-\frac{x}{L}\right)+\frac{q_g L^2}{2k(T_1-T_2)}\frac{x}{L}\left(1-\frac{x}{L}\right)$$

$$\frac{T-T_2}{T_1-T_2}=\left(1-\frac{x}{L}\right)\left[1+\frac{q_g L^2}{2k(T_1-T_2)}\frac{x}{L}\right]$$

$$\frac{T-T_2}{T_1-T_2}=\left(1-\frac{x}{L}\right)\left(1+\frac{Bx}{L}\right) \tag{20}$$

Where $B=\dfrac{q_g L^2}{2k(T_1-T_2)}$

Equation (20) represents temperature distribution equation in the slab having a heat generating source present inside it. In order to find out the location of maximum temperature in the slab equation (20) is differentiated with respect to 'x' and equated equal to zero.

$$\frac{d}{dx}\left(\frac{T-T_2}{T_1-T_2}\right)=\frac{d}{dx}\left[\left(1-\frac{x}{L}\right)\left(1+\frac{Bx}{L}\right)\right]$$

$$\frac{dT}{dx}=\frac{d}{dx}\left[\left(1+\frac{Bx}{L}-\frac{x}{L}-\frac{Bx^2}{L^2}\right)\right]$$

$$0=\left[\left(\frac{B}{L}-\frac{1}{L}-\frac{2Bx}{L^2}\right)\right]$$

$$0=\frac{1}{L}\left(B-1-\frac{2Bx}{L}\right)$$

$$0=\left(B-1-\frac{2Bx}{L}\right)$$

$$\frac{2Bx}{L}=(B-1)$$

$$\frac{x}{L}=\frac{B-1}{2B} \tag{21}$$

Equation (21) gives the location of maximum temperature in the slab. The equation representing the maximum value of temperature is obtained by substituting the value of maximum x /L from equation (21) into equation (20).

$$\frac{T_{max}-T_2}{T_1-T_2}=\left(1-\frac{B-1}{2B}\right)\left(1+\frac{B(B-1)}{2B}\right)$$

$$\frac{T_{max}-T_2}{T_1-T_2}=\left(\frac{2B-B+1}{2B}\right)\left(\frac{2+B-1}{2}\right)$$

$$\frac{T_{max}-T_2}{T_1-T_2}=\frac{(B+1)^2}{4B} \qquad (22)$$

Flow of heat from one surface is given as

$$Q_1=-kA\left(\frac{dT}{dx}\right)_{x=0}$$

Flow of heat from one surface is given as

$$Q_1=-kA\left(\frac{dT}{dx}\right)_{x=0}$$

From equation (14) substituting the value of dT/dx, we get

$$Q_1=-kA\left(C_1-\frac{q_g}{k}x\right)_{x=0}$$

Substituting the value of $C_1$ from equation (17), we get

$$Q_1=-kA\left(\frac{T_2-T_1}{L}+\frac{q_g}{2k}L-\frac{q_g}{k}x\right)_{x=0}$$

$$Q_1=kA\left(\frac{T_1-T_2}{L}-\frac{q_g}{2k}L\right)$$

Similarly heat flow from the other surface

$$Q_2 = -kA\left(\frac{dT}{dx}\right)_{x=L}$$

$$Q_2 = -kA\left(\frac{T_2 - T_1}{L} + \frac{q_g}{2k}L - \frac{q_g}{k}x\right)_{x=L}$$

$$Q_2 = -kA\left(\frac{T_2 - T_1}{L} + \frac{q_g}{2k}L - \frac{q_g}{k}L\right)$$

$$Q_2 = kA\left(\frac{T_1 - T_2}{L} - \frac{q_g}{2k}L + \frac{q_g}{k}L\right)$$

$$Q_2 = kA\left(\frac{T_1 - T_2}{L} + \frac{q_g}{2k}L\right)$$

In case maximum temperature occurs inside the slab, heat will flow from both surfaces of the slab and total heat flow will be given as:

$$Q_T = Q_1 + Q_2$$

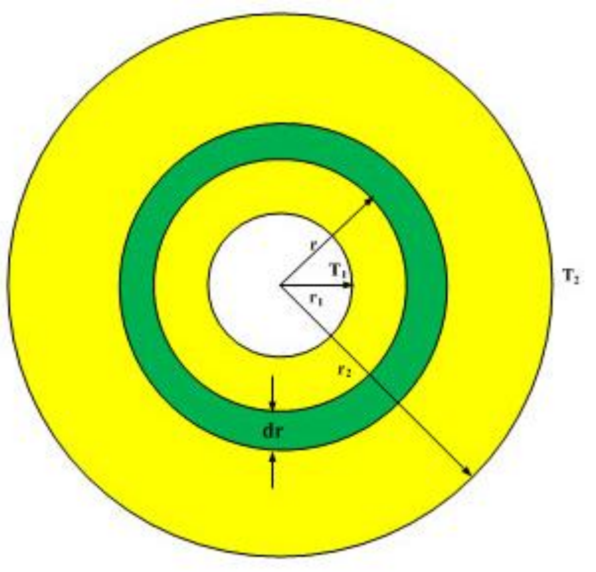
In case $T_1$ is the maximum temperature, heat will flow towards x (+ve only) and heat lost will be given as:

$$Q_T = Q_2$$

**<u>One-Dimensional Heat Flow through a cylinder with Heat Generation</u>**

**i) A hollow Cylinder:**

Consider a hollow cylinder of length L having inner and outer radii $r_1$ and $r_2$ respectively in which flow of heat is unidirectional along the radial direction. $T_1$ and $T_2$ are temperatures of the inner and outer surfaces of the cylinder respectively. In order to determine temperature distribution and heat flow rate, a small element at radius r and thickness dr is considered. A heat source present inside the strip is generating $q_g$ amount of heat per unit volume as shown in Figure 3.



Heat conducted into the element, $Q_r = -k(2\,\Pi\, r\, L)\, dT/dr$ (23)

Heat generated in the element, $Q_g = 2\,\Pi\, r\, L\, dr\, q_g$ (24)

Heat conducted out of the element, $$Q_{r+dr} = Q_r + \frac{d}{dr}(Q_r)\,dr \qquad (25)$$

For steady state condition of heat flow

Heat conducted into the element + Heat generated in the element = Heat conducted out of the element

$$Q_r + Q_g = Q_{r+dr}$$

$$Q_r + Q_g = Q_r + \frac{d}{dr}(Q_r)\,dr$$

$$Q_g = \frac{d}{dr}(Q_r)\,dr \qquad (26)$$

Substituting the values of $Q_r$ and $Q_g$ from equations (23 and 24) in equation (26), we get

$$2\pi r L dr q_g = \frac{d}{dr}\left(-2\pi r L k\frac{dT}{dr}\right)dr$$

$$r q_g = -k\frac{d}{dr}\left(r\frac{dT}{dr}\right)$$

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \frac{r q_g}{k} = 0 \qquad (27)$$

In order to find out the solution of the above equation, integrate it with respect to r

$$r\frac{dT}{dr} + \frac{r^2 q_g}{2k} = C_1 \qquad (28)$$

$$\frac{dT}{dr} + \frac{r q_g}{2k} = \frac{C_1}{r} \qquad (29)$$

Integrating equation (4.28) again with respect to r, we get

$$T + \frac{q_g}{4k}r^2 = C_1 \log_e r + C_2 \qquad (30)$$

$C_1$ and $C_2$ are constants of integration and the expressions for these constants can be found out by using the following boundary conditions

At $r=r_1$, $T=T_1$ and at $r=r_2$, $T=T_2$

$$T_1 + \frac{q_g}{4k} r_1^2 = C_1 \log_e r_1 + C_2 \tag{31}$$

$$T_2 + \frac{q_g}{4k} r_2^2 = C_1 \log_e r_2 + C_2 \tag{32}$$

Subtracting equation (32) from equation (31), we get

$$T_1 - T_2 + \frac{q_g}{4k}\left(r_1^2 - r_2^2\right) = C_1 \log_e \frac{r_1}{r_2}$$

$$C_1 \log_e \frac{r_2}{r_1} = \frac{q_g}{4k}\left(r_2^2 - r_1^2\right) - \left(T_1 - T_2\right)$$

$$C_1 = \frac{\frac{q_g}{4k}\left(r_2^2 - r_1^2\right) - \left(T_1 - T_2\right)}{\log_e \frac{r_2}{r_1}} \tag{33}$$

Substituting the value of $C_1$ in equation (31), we get

$$T_1 + \frac{q_g}{4k} r_1^2 = \frac{\frac{q_g}{4k}\left(r_2^2 - r_1^2\right) - \left(T_1 - T_2\right)}{\log_e \frac{r_2}{r_1}} \log_e r_1 + C_2$$

$$C_2 = T_1 + \frac{q_g}{4k} r_1^2 - \frac{\frac{q_g}{4k}\left(r_2^2 - r_1^2\right) - \left(T_1 - T_2\right)}{\log_e \frac{r_2}{r_1}} \log_e r_1 \tag{34}$$

Substituting the values of $C_1$ and $C_2$ in equation (30), we get

$$T + \frac{q_g}{4k} r^2 = \frac{\frac{q_g}{4k}\left(r_2^2 - r_1^2\right) - \left(T_1 - T_2\right)}{\log_e \frac{r_2}{r_1}} \log_e r + T_1 + \frac{q_g}{4k} r_1^2 - \frac{\frac{q_g}{4k}\left(r_2^2 - r_1^2\right) - \left(T_1 - T_2\right)}{\log_e \frac{r_2}{r_1}} \log_e r_1$$

$$T + \frac{q_g}{4k} r^2 = T_1 + \frac{q_g}{4k} r_1^2 + \frac{\frac{q_g}{4k}\left(r_2^2 - r_1^2\right) - \left(T_1 - T_2\right)}{\log_e \frac{r_2}{r_1}} \left(\log_e r - \log_e r_1\right)$$

$$T - T_1 = \frac{q_g}{4k}\left(r_1^2 - r^2\right) + \frac{\frac{q_g}{4k}\left(r_2^2 - r_1^2\right) - \left(T_1 - T_2\right)}{\log_e \frac{r_2}{r_1}} \left(\log_e \frac{r}{r_1}\right) \tag{35}$$

Dividing both sides of equation (35) by $T_1 - T_2$, we get

$$\frac{T-T_1}{T_1-T_2}=\frac{q_g}{4k(T_1-T_2)}\left(r_1^2-r^2\right)+\frac{\frac{q_g}{4k}\left(r_2^2-r_1^2\right)-(T_1-T_2)}{(T_1-T_2)\log_e\frac{r_2}{r_1}}\left(\log_e\frac{r}{r_1}\right)$$

$$\frac{T-T_1}{T_1-T_2}=\frac{q_g\left(r_1^2-r^2\right)}{4k(T_1-T_2)}+\left(\frac{\frac{q_g}{4k}\left(r_2^2-r_1^2\right)}{(T_1-T_2)}-1\right)\frac{1}{\log_e\frac{r_2}{r_1}}\left(\log_e\frac{r}{r_1}\right)$$

$$\frac{T-T_1}{T_1-T_2}=\frac{q_g\left(r_1^2-r^2\right)}{4k(T_1-T_2)}+\frac{\frac{q_g}{4k}\left(r_2^2-r_1^2\right)}{(T_1-T_2)}\frac{\log_e\frac{r}{r_1}}{\log_e\frac{r_2}{r_1}}-\frac{\log_e\frac{r}{r_1}}{\log_e\frac{r_2}{r_1}}$$

$$\frac{T-T_1}{T_1-T_2}=\frac{q_g}{4k(T_1-T_2)}\left[\left(r_1^2-r^2\right)+\left(r_2^2-r_1^2\right)\frac{\log_e\frac{r}{r_1}}{\log_e\frac{r_2}{r_1}}\right]-\frac{\log_e\frac{r}{r_1}}{\log_e\frac{r_2}{r_1}}$$

Multiplying and diving Right Hand Side of the above equation by $r^2$, we get

$$\frac{T-T_1}{T_1-T_2}=\frac{q_g r^2}{4k(T_1-T_2)}\left[\left(\frac{r_1^2}{r_2^2}-\frac{r^2}{r_2^2}\right)+\left(1-\frac{r_1^2}{r_2^2}\right)\frac{\log_e\frac{r}{r_1}}{\log_e\frac{r_2}{r_1}}\right]-\frac{\log_e\frac{r}{r_1}}{\log_e\frac{r_2}{r_1}} \qquad (36)$$

Equation (36) represents temperature distribution inside a hollow cylinder with heat generation.

**ii) A Solid Cylinder**

In case of solid cylinder, the governing equation remains same as equation (30)

$$T+\frac{q_g}{4k}r^2=C_1\log_e r+C_2$$

Differentiating above equation with respect to r, we get

$$\frac{dT}{dr}+\frac{rq_g}{2k}=\frac{C_1}{r}$$

Applying the boundary conditions

At r=0, $\frac{dT}{dr} = 0$, so $C_1=0$

At $r=r_2$, $T=T_2$,

$$C_2 = T_2 + \frac{q_g}{4k} r_2^2$$

Substituting the values of $C_1$ and $C_2$ in equation (30), we get

$$T + \frac{q_g}{4k} r^2 = T_2 + \frac{q_g}{4k} r_2^2$$

$$T = T_2 + \frac{q_g r_2^2}{4k}\left[1 - \left(\frac{r}{r_2}\right)^2\right]$$

$$T - T_2 = \frac{q_g r_2^2}{4k}\left[1 - \left(\frac{r}{r_2}\right)^2\right] \quad (37)$$

Equation (37) represents temperature distribution equation in a solid cylinder with heat generation. Maximum temperature will occur at $r=r_2$, and will be expressed as

$$T_{max} = T_2 + \frac{q_g r_2^2}{4k}$$

$$T_{max} - T_2 = \frac{q_g r_2^2}{4k} \quad (38)$$

Dividing equation (37) by equation (38), we get

$$\frac{T - T_2}{T_{max} - T_2} = \left[1 - \left(\frac{r}{r_2}\right)^2\right] \quad (39)$$

Heat flow through a solid cylinder is expressed as

$$Q = \left[ -kA \frac{dT}{dr} \right]_{r=r_2}$$

$$Q = -k2\pi r_2 L \frac{d}{dr} \left[ T_2 + \frac{q_g r_2^2}{4k} \left\{ 1 - \left( \frac{r}{r_2} \right)^2 \right\} \right]_{r=r_2}$$

$$Q = -k2\pi r_2 L \frac{d}{dr} \left[ \frac{q_g}{4k} \{-2r\} \right]_{r=r_2}$$

$$Q = \frac{4\pi k r_2^2 L q_g}{4k}$$

$$Q = \pi r_2^2 L q_g \qquad (40)$$

Heat conducted = Volume of cylinder x heat generating capacity per unit volume per unit Time

For steady state conditions, heat conducted at $r = r_2$ must be equal to heat convected from outer surface of cylinder to the surrounding fluid.

Heat Conducted = Heat convected

From equation (40), we can write

$$\pi r_2^2 L q_g = h \times 2\pi r_2 L \left( T_2 - T_f \right)$$

$T_f$ is temperature of fluid surrounding the cylinder.

$$T_2 = T_f + \frac{r_2 q_g}{2h}$$

Substituting the value of $T_2$ in equation (37), we get

$$T = T_f + \frac{r_2 q_g}{2h} + \frac{r_2^2 q_g}{4k} \left[ 1 - \left( \frac{r}{r_2} \right)^2 \right]$$

**<u>One-Dimensional Heat Flow through a sphere with Heat Generation</u>**

Consider steady state heat conduction through a hollow sphere having $r_1$ and $r_2$ as inner and outer radii respectively. Temperature of the inner and outer surfaces is $T_1$ and $T_2$ respectively. Heat is flowing from inner to outer surface as $T_1$ is greater than $T_2$ as shown in Figure 4.

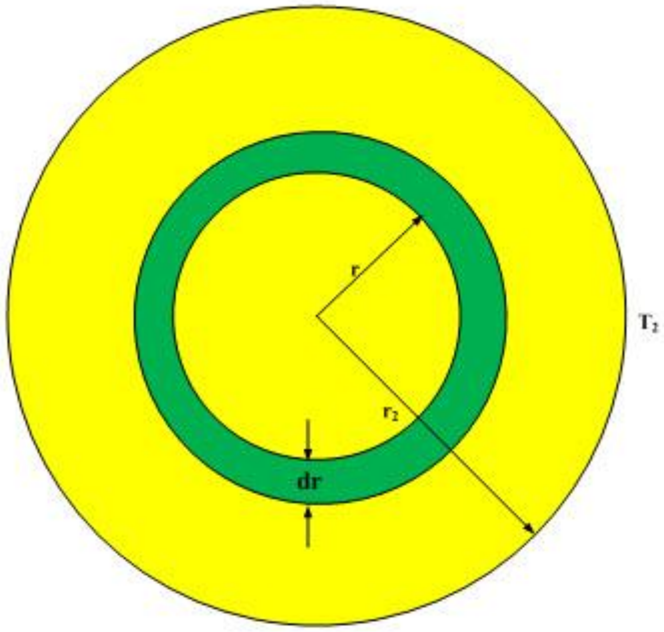



Figure 4

The general conduction equation which governs the conduction heat transfer is written as

$$\left[\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 T}{\partial \phi^2}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right)+\frac{q_g}{k}\right]=\frac{1}{\alpha}\frac{\partial T}{\partial t}dt \tag{41}$$

Since it is a case of one-dimensional, stead heat conduction through a wall of uniform conductivity with heat generation, therefore, $\frac{dT}{dt}=0$ and $\frac{dT}{d\phi}=\frac{dT}{d\theta}=0$

Therefore, equation (41) reduces to

$$\frac{d^2T}{dr^2}+\frac{2}{r}\frac{dT}{dr}+\frac{q_g}{k}=0 \tag{42}$$

The above equation can be written as

$$r\frac{d^2T}{dr^2}+\frac{dT}{dr}+\frac{dT}{dr}+\frac{q_g}{k}r=0$$

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{dT}{dr}+\frac{q_g}{k}r=0 \tag{43}$$

Integrating equation (43) with respect to r, we get

$$r\frac{dT}{dr}+T+\frac{q_g}{k}\frac{r^2}{2}=C_1$$

$$\frac{d}{dr}(rT)+\frac{q_g}{k}\frac{r^2}{2}=C_1$$

Upon integrating above equation once more with respect to r, we get

$$rT+\frac{q_g}{k}\frac{r^3}{6}=C_1r+C_2 \tag{44}$$

Applying the first boundary condition i.e. at $r = 0$, $dT/dr = 0$ to equation (44), we get

$$C_2 = 0 \tag{45}$$

Applying the second boundary condition i.e at $r = r_2$, $T = T_2$ to equation (43), we get

$$r_2T_2+\frac{q_g}{k}\frac{r_2^3}{6}=C_1r_2$$

$$T_2+\frac{q_g}{k}\frac{r_2^2}{6}=C_1 \tag{46}$$

Substituting the values of $C_1$ and $C_2$ in equation (44), we get

$$rT+\frac{q_g}{k}\frac{r^3}{6}=r\left(T_2+\frac{q_g}{k}\frac{r_2^2}{6}\right)$$

$$T+\frac{q_g}{k}\frac{r^2}{6}=\left(T_2+\frac{q_g}{k}\frac{r_2^2}{6}\right)$$

$$T=T_2+\frac{q_g}{k}\frac{r_2^2}{6}\left(1-\left(\frac{r}{r_2}\right)^2\right) \tag{47}$$

Equation (47) represents temperature distribution equation in a solid sphere having a heat source present inside it.

Heat flow rate through a sphere with heat generation can be determined by using the following equation

$$Q = -kA\left(\frac{dT}{dr}\right)_{r=r_2}$$

$$Q = -k4\pi r_2^2 \frac{d}{dr}\left(T_2 + \frac{q_g}{6k}(r_2^2 - r^2)\right)_{r=r_2}$$

$$Q = -4\pi k r_2^2\left(\frac{q_g}{6k}(-2r)\right)_{r=r_2}$$

$$Q = 4\pi k r_2^2 \frac{q_g}{3k} r_2$$

$$Q = \frac{4}{3}\pi r_2^3\, q_g$$

Heat conducted = Volume of sphere x heat generating capacity

For steady state conditions, heat conducted through a sphere must be equal to heat convected from outer surface of the sphere

$$\frac{4}{3}\pi r_2^3\, q_g = 4\pi r_2^2 h(T_2 - T_f)$$

$$T_2 = T_f + \frac{q_g r_2}{3h}$$

Substitute the value of $T_2$ from above equation in equation (47), we get

$$T = T_2 + \frac{q_g r_2}{3h} + \frac{q_g r_2^2}{6k}\left(1 - \left(\frac{r}{r_2}\right)^2\right)$$

**Lesson 5. Electrical analogy and Numerical Problems related to conduction**

**Electrical Analogy For Conduction Problems**

Consider heat flowing through a slab of thickness 'L' and area "A' and $T_1$ and $T_2$ are the temperatures on the two faces of the slab as shown in Figure 1. Heat transfer from high temperature side to low temperature side is expressed as

$$Q = \frac{kA}{L}(T_1 - T_2) \qquad (1)$$

Where ($T_1$ –$T_2$) is the thermal potential

$\frac{kA}{L}$ is the thermal resistance

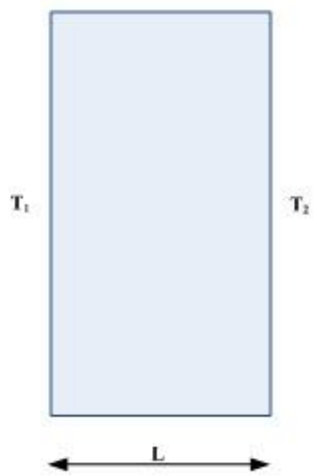


Figure 1

Now, consider an electric circuit having resistance 'R' and electric potential $E_1$ and $E_2$ at the ends as shown in Figure 2. Current 'I' passing through the circuit can be expressed as

$$I = \frac{(E_1 - E_2)}{R} \qquad (2)$$

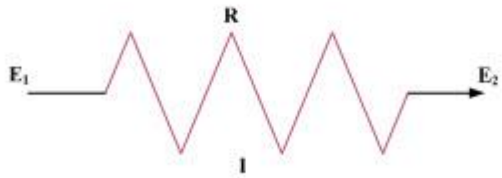


Figure 2

Equations (1) and (2) are found to be symmetrical on comparison. 'Q' amount of heat flows through the slab having thermal resistance when a thermal potential ($T_1$ –$T_2$) exits. Similarly, 'I' amount of current passes through the circuit having resistance 'R' when an electric potential $(E_1 - E_2)$ exists. Therefore, flow of heat through the slab can be represented by an electric circuit as shown in Figure 3.

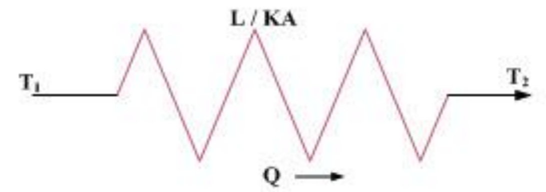



Figure 3

If a hot gas at temperature $T_g$ is in contact with one side of the slab and air at temperature $T_a$ at the other side, then heat transfer from the hot gas to air through this slab of thickness can be represented by an electric circuit as shown in Figure 4

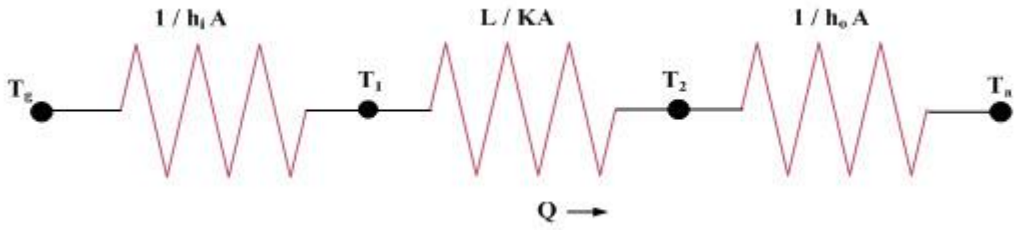



Figure 4

Heat transfer from hot gas to air can be expressed as

$$Q = \frac{(T_g - T_a)}{1/h_i A + L/KA + 1/h_0 A} \tag{3}$$

For a slab made of three material having thermal conductivities $K_A$, $K_B$ and $K_C$ respectively and is exposed to a hot gas on one side and atmospheric air on the other side as shown in Figure 5, an equivalent electric circuit has been shown in Figure 6. Heat transfer from the hot gas to atmospheric air is expressed as

$$Q = \frac{(T_g - T_a)}{1/h_i A + L_1/K_A A_A + L_2/K_B A_B + L_3/K_C A_C + 1/h_0 A} \tag{4}$$

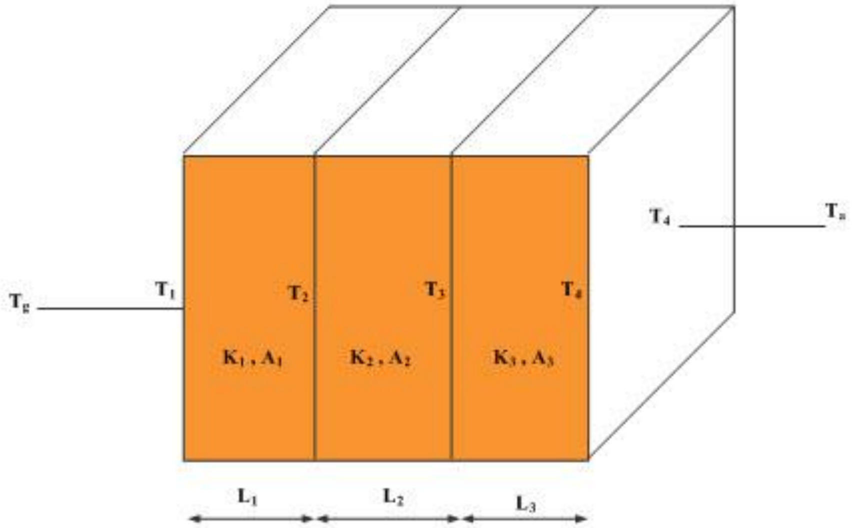



Figure 5


Figure 6

Similarly for the composite slab shown in Figure 7, an equivalent electric circuit has been shown in Figure 8.

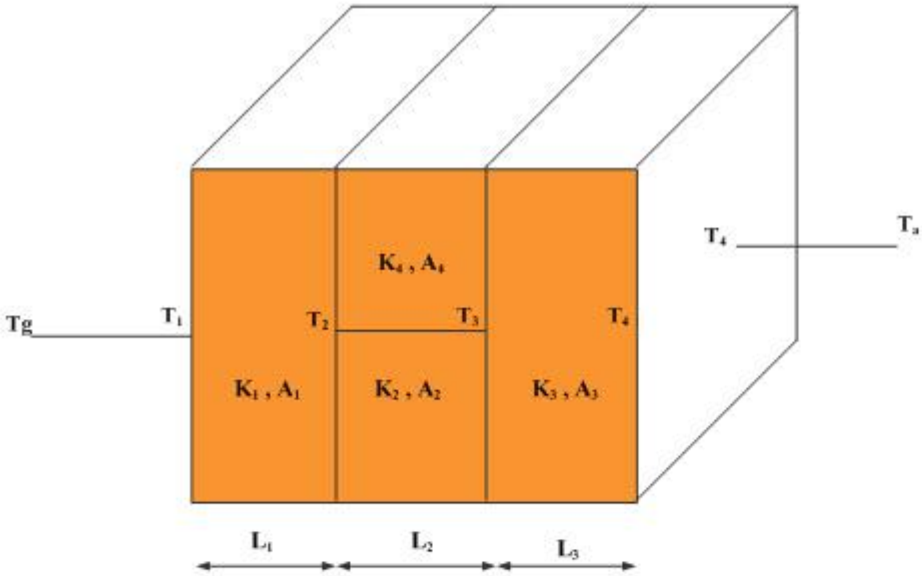



Figure 7

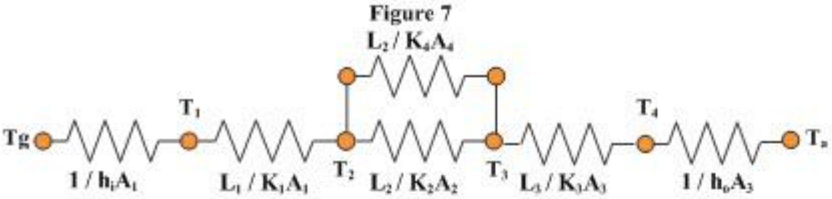



Figure 8

**Example 3.14 The walls of heating unit in cold region comprise three layers**

**10cm outer brick work (k =0.75 W/m-deg)**

**1.5cm inner wooden paneling (k =0.75 W/m-deg)**

**8cm intermediate layer of insulating material**

**The insulation layer is stated to offer resistance twice the thermal resistance of brick work. If the inside and outside temperatures of the composite wall are 30°C and -10°C respectively, determine the rate of heat loss per unit area of the wall and the thermal conductivity of the insulating material.**

**Solution:** Thermal resistance for a plane wall of thickness δ, area A and thermal conductivity k is prescribed by the relation $R_t = \delta/kA$

Resistance of brick work, $R_{t_1} = \frac{10\times10^{-2}}{0.75\times1} = 0.133 deg/W$

Resistance of wooden paneling, $R_{t_2} = \frac{1.5\times10^{-2}}{0.133\times1} = 0.1128 deg/W$

Resistance of insulating material, $R_{t_3} = 2\times0.133 = 0.266 deg/W$

Total resistance $R_t = \sum R_t = 0.133+0.1128+0.266 = 0.5118$ deg/W

$\therefore$ Heat loss $Q = \frac{\Delta t}{\sum R_t} = \frac{30-(-10)}{0.5118} = 78.155\ W$

(b) Thermal conductivity of insulating material,

$$k = \frac{\delta}{A\,R_t} = \frac{7.5\times10^{-2}}{1\times0.5118} = \frac{0.1465W}{m} - deg$$

**Example 3.15 A furnace wall is a made up of steel plate 10 mm thick (k=62.8 kj/m-hr-deg) lined on inside with silica briks 150 mm thick (k =7.32 kj/m-hr-deg) and on the outside with magnesia bricks 200 mm thick (k = 18.84 kj/m-hr-deg). The inside and outside surfaces of the wall are at temperature 650° C respectively. Make calculations for the heat loss from unit area of the wall.**

**It is required that the heat loss be reduced to 10Mj/hour by means of air gap between steel and magnesia bricks. Estimate the necessary width of air gap if thermal conductivity for is 0.126 kj/m-hr-deg.**

**Solution:** Thermal resistance for a plane wall of thickness δ, area A and thermal conductivity k is prescribed by the relation $R_t = \delta/kA$.

$\therefore$ Resistance of steel plate $R_{t_1} = \frac{0.01}{62.8\times1} = 0.000159 \deg hr/kj$

Resistance of silica bricks $R_{t_2} = \frac{0.15}{7.32\times1} = 0.02049 \deg hr/kj$

Resistance of magnesia bricks $R_{t_3} = \frac{0.20}{18.84\times1} = 0.01061 \deg hr/kj$

Total resistance of the composite wall,

$$\sum R_t = R_{t_1} + R_{t_2} + R_{t_3}$$
$$= 0.03126 \text{ deg hr/kj}$$

Heat loss from the wall $= \frac{\Delta t}{\sum R_t} = \frac{650-125}{0.03126} = 16795 \frac{kj}{hr}$

To reduce the heat loss to 10 Mj/hr, the thermal resistance should be increased to:

$$\frac{650-125}{10\times10^3} = 0.0525 \deg hr/kj$$

$\therefore$ Thermal resistance for the air gap,

$$\frac{\delta}{kA} = 0.0525 - 0.03126 = 0.02124 \deg hr/kj$$

$\therefore$ Thermal of air gap, $\delta = 0.02124 \times 10^{-3}$ m $=$ **2.676 mm**

**Example 3.16 A furnace wall comprises three layers: 13.5 cm thick inside layer of fire brick, 7.5 cm thick middle layer of insulating brick and 11.5 cm thick outside layer of red brick. The furnace operates at 870°C and it is anticipated that the outside of this composite wall can be maintained at 40°C by the circulation of air. Assuming close bonding of layers at their interfaces, find the rate of heat loss from the furnace and the wall interface temperature. The wall measures 5m2m and the data on thermal conductivities is:**

**Fire brick $k_1$ = 2.4 W/m-deg**

**Insulating brick $k_2$ = 0.14 W/m-deg**

**Red brick $k_3$ = 0.85 W/m-deg**

**Solution:** The wall area (5m 2m) = 10 m² is constant for all layers of the composite wall. The thermal resistance $R_t$ of a slab (thickness δ, conductivity k and area A ) is given by

$$R_t = \frac{\delta}{kA}$$

$\therefore$ Resistance of fire brick $R_{t_1} = \frac{0.135}{2.4\times10} = 0.00562\,\deg/W$

$\therefore$ Resistance of insulating brick $R_{t_2} = \frac{0.075}{0.14\times10} = 0.05357\,\deg/W$

$\therefore$ Resistance of red brick $R_{t_{13}} = \frac{0.115}{0.85\times10} = 0.01353\,\deg/W$

Total resistance of the composite wall,

$$\sum R_t = R_{t_1} + R_{t_2} + R_{t_3} = 0.07272\ \deg/W$$

The heat will flow from the inside of the furnace (temperature $t_1 = 870°$ C) to outside of the composite wall (temperature $t_4 = 40°C$)

$$\text{Heat loss} = \frac{t_1 - t_4}{\sum R_t} = \frac{870 - t_2}{0.07272} = \mathbf{11413.64W}$$

(b) Since heat flowing through each layer is same, then for inside layer of fire brick

$$10593.5 = \frac{t_1 - t_2}{.005625} = \frac{870 - t_2}{0.005625}$$

Where $t_2$ is the temperature at the interface of fire brick and insulating brick.

$$t_2 = 870 - 10593.5 \times 0.005625$$
$$= 810.82°C$$

Similarity for the mid layer of insulating brick,

$$10593.5 = \frac{t_2 - t_3}{0.05357} = \frac{810.82 - t_3}{0.05357}$$

Where $t_3$ is the temperature at the interface of insulating brick and the red brick

$$t_3 = 810.82 - 10593.5 \times 0.05357 = 243.33°C$$

**Check :** The temperature $t_3$ could also be worked out by considering the heat flow through the outside layer of red brick.

$$10593.5 = \frac{t_3 - t_4}{0.01357} = \frac{t_3 - 40}{0.01357}$$

$$t_3 = 40 + 10593.5 \times 0.01357 = 183.75°C$$

## Lesson 6. Numericals on conduction

**Example 3.20 Determine the heat transfer rate across a composite slab which is made of different materials with top and bottom as shown in fig. 3.16. The entire left-hand face is held at the temperature $T_1$ while the entire right hand face is at the temperature $T_2$. The conductivities of the two different materials are stated as $k_a$ and $k_b$, and their areas as viewed in the direction of slab thickness δ are $A_a$ and $A_b$ respectively. Steady state exists, there is no heat generation and the slab is so thin that any edge effects can be neglected. Interpret the result in terms of an electrical circuit.**

**Solution:** Applying Fourier law of heat conduction separately to materials a and b, we obtain

$$Q_a = \frac{k_a A_a (T_1 - T_2)}{\delta}$$

$$Q_b = \frac{k_b A_a (T_1 - T_2)}{\delta}$$

The desired quantity of heat transfer through the slab equal the sum of $Q_a$ and $Q_b$

$$Q = Q_a + Q_b$$

$$= \frac{k_a A_a (T_1 - T_2)}{\delta} + \frac{k_b A_a (T_1 - T_2)}{\delta}$$

$$= \left\{\frac{1}{\delta/(k_a A_a)} + \frac{1}{\delta/(k_b A_b)}\right\} (T_1 - T_2)$$

$$= \left(\frac{1}{R_{t_a}} + \frac{1}{R_{t_b}}\right) (T_1 - T_2) = \frac{1}{R_t} (T_1 - T_2)$$

Apparently, the two thermal resistances

$$R_{t_a} = \frac{\delta}{k_a A_a} \quad \text{and} \quad R_{t_b} = \frac{\delta}{k_b A_b}$$

Appear in the same way as two electrical resistors in parallel. Accordingly the electrical circuit for the heat transfer through the given composite slab will be as indicated in fig. 3.16.

**Example 3.20 Find the heat flow rate through the composite wall as shown in fig.3.17. Assume one-dimensional flow and take**

**$K_a$ =150 W/m-deg; $k_b$ = 30 W/m-deg; $k_c$ = 65 W/m-deg; $k_d$ = 50 W/m-deg**

**Solution:** The equivalent thermal circuit for heat flow in the composite system has been shown in fig. 3.18.

$$R_a = \frac{\delta_a}{k_a A_a} = \frac{0.03}{150 \times (0.1 \times 0.1)} = 0.02^\circ C/W$$

$$R_b = \frac{\delta_b}{k_b A_b} = \frac{0.08}{30 \times (0.1 \times 0.03)} = 0.089^\circ C/W$$

$$R_c = \frac{\delta_c}{k_c A_c} = \frac{0.08}{65 \times (0.1 \times 0.07)} = 0.176^\circ C/W$$

$$R_d = \frac{\delta_d}{k_d A_d} = \frac{0.05}{50 \times (0.1 \times 0.1)} = 0.01^\circ C/W$$

The resistances $R_b$ and $R_c$ are in parallel and their equivalent resistance $R_{eq}$ is

$$R_{eq} = \frac{R_b \times R_c}{R_b + R_c} = \frac{0.89 \times 0.176}{0.89 + 0.176} = 0.1469^\circ C/W$$

The equivalent resistance is now in series with resistance $R_a$ and $R_d$. The total thermal resistance for the entire circuit then becomes

$$\sum R_t = R_a + R_{eq} + R_b$$

$$= 0.02 + 0.1469 + 0.1 = 0.2669^\circ C/W$$

Hence heat transfer rate through the system is

$$Q = \frac{\Delta t}{\sum Rt} = \frac{400 - 60}{0.2669} = \mathbf{1273.88W}$$

**Example 3.24 Two slabs, each 200 mm thick and made of materials with thermal conductivities of 16 W/m-deg and 1600 W/m-deg, are placed in contact which is not perfect. Due to roughness of surfaces, only 40% of area is in contact and air fills 0.02 mm thick gap in the remaining area. If the extreme surfaces of the arrangement are at temperatures of 250° C and 30° C, determine the heat flow through the composite system, the contact resistance and temperature drop in contact.**

**Take thermal conductivity of air as 0.032 W/m-deg and assume that half of the contact (of the contact area) is due to either metal.**

**Solution:** Refer fig. 3.22 for the composite system and its equivalent thermal resistance

The various thermal resistances to flow of heat are:

i. $R_{t_a} = \frac{\delta_a}{k_a A_a} = \frac{200\times10^{-3}}{16\times1} = 0.01250\ deg/W$

ii. $R_{t_b} = \frac{\delta_b}{k_b b} = \frac{0.02\times10^{-3}}{16\times0.2} = 0.00000625\ deg/W$

iii. $R_{t_c} = \frac{\delta_c}{k_c A_c} = \frac{0.02\times10^{-3}}{0.032\times0.6} = 0.001042\ deg/W$

iv. $R_{t_d} = \frac{\delta_d}{k_d A_d} = \frac{0.02\times10^{-3}}{200\times0.2} = 0.0000005\ deg/W$

v. $R_{t_e} = \frac{\delta_e}{k_e A_e} = \frac{100\times10^{-3}}{200\times1} = 0.0005\ deg/W$

The resistances , and are in parallel and their equivalent resistance $(R_t)_{eq}$ is

$$\frac{1}{(R_t)_{eq}} = \frac{1}{R_{t_b}} + \frac{1}{R_{t_c}} + \frac{1}{R_{t_d}}$$

$$= \frac{1}{0.00000625} + \frac{1}{0.001042} + \frac{1}{0.0000005}$$

$$= 160000+959.7+2000000=2160959.7$$

$$\therefore (R_t)_{eq} = \frac{1}{2160959.7} = 0.462\times10^{-6} deg/W$$

This equivalent resistance is now in series with resistance and . The total thermal resistance for the entire circuit then becomes

$$\sum R_t = 0.01250+0.462\times10^{-6} + 0.0005$$
$$=0.01300\ deg/W$$

Hence, heat transfer rate through the system is

$$Q = \frac{\Delta t}{\sum R_t} = \frac{250-30}{0.01300} = 16923.07\ W$$

(b) Contact resistance $= 0.462\times10^{-6} deg/W$

Temperature drop in contact $=$ Q$\times$contact resistance

$$=16923.07\times(0.462\times10^{-6}) = 0.007818℃$$

**Example 3.34 A glazed window, made of 8mm thick glass of thermal conductivity 1.5 W/mK, has its outside surface maintained at 5°C so that frosting is reduced. The surroundings are at-10°C with convective coefficient 55 W/m²K.the desired condition is attained by providing a uniform heat flux at the inner surface of the window which is fitted into a room where the air temperature is 25° with convection of 12.5 W/m²K. Make calculations for the heating required per m² area.**

**Solution:** Refer fig.3.35 for the window fixture with specified data and thermal circuit for the resistance involved.

Let $t_1$ be the temperature at the heater. Under steady state conditions heat conducted through the glass barrier equals the heat convected through the outde film. That is

$$\frac{kA(t_1-t_2)}{\delta} = h_0A(t_2-t_0)$$

Considering unit area,

$$\frac{1.5\times1(t_1-t_2)}{\delta} = 55\times 1[5-(-10)]=825 \text{ W}$$

That gives:

(b) From energy balance

Heat flow (Q) + heat received by convection fram room ($Q_2$)

= heat conducted through the glass barrier ($Q_1$)

or heat flux $Q = Q_1 - h_i A (t_i - t_1)$

$=825-12.51(25-9.4)=$ **630 W**

Thus, the heat required per m² area is 630W.

**Example 3.34 A square plane heater of 0.8 kW rating and measuring 15cm15cm is placed between two slabs A and B and the following data refers to these slabs:**

**Slab A is 1.8 cm thick with k = 55 W/m-deg**

**Slab B is 1cm thick with k = 0.2 W/m-deg**

**The outside heat transfer coefficients on the side of plate A and B are 200 W/m²-deg and 45 W/m²-deg respectively. If the surrounding environment is at 27°C temperature, make calculations for the maximum temperature of the system and outside surface temperature of both slabs.**

**Solution:** Refer fig. 3.36 for the arrangement and thermal resistance network for the system.

The individual resistances are evaluated as:

$$R_{t1} = \frac{\delta_1}{k_1 A_1} = \frac{1.8 \times 10^{-2}}{55 \times (0.15 \times 0.15^{-2})^2} = 0.145\ K/W$$

$$R_{t2} = \frac{1}{h_1 A_1} = \frac{1}{200 \times (0.15 \times 0.15^{-2})} = 0.222\ K/W$$

These resistances are in series and accordingly for slab A (left branch of the circuit)

$$R_{t1} + R_{t2} = 0.0145 + 0.222 = 0.2365 K/W$$

$$R_{t3} = \frac{\delta_2}{k_2 A} = \frac{1 \times 10^{-2}}{0.2 \times (0.15 \times 0.10^{-2})^2} = 2.222\ K/W$$

$$R_{t4} = \frac{1}{h_2 A_2} = \frac{1}{45 \times (15 \times 0.15^{-2})^2} = 0.987\ K/W$$

These resistances are in series and accordingly for slab B (right branch of the circuit)

$$R_{t3} + R_{t4} = 0.222 + 0.987 = 3.209\ K/W$$

(a) Rating of Heater, $Q = Q_A + Q_B$

$$= \frac{T_{max} - T_a}{R_{t1} + R_{t2}} + \frac{T_{max} - T_a}{R_{t3} + R_{t4}}$$

$$= (T_{max} - T_a)\left[\frac{1}{0.2365} + \frac{1}{3.209}\right] = 4.5396(T_{max} - T_a)$$

Maximum temperature in the system,

$$T_{max} = \frac{Q}{4.5396} + T_a = \frac{0.8 \times 10^3}{4.5396} + 27 = 203.24°C$$

(b) Considering left side branch of the circuit (slab A)

$$Q_A = \frac{T - T_a}{R_{t1} + R_{t2}} = \frac{203.24 - 27}{0.2365} = 745.2W$$

If $T_1$ is the temperature at exposed surface of slab A, then

$$Q_A = \frac{T_1 - T_a}{R_{t2}} \quad ; \quad T_1 = Q_A R_{t2} + T_a$$

$$= 745.2 \times 0.222 + 27 = 192.43°C$$

Considering right side branch of the circuit (slab B)

$$Q_B = \frac{T_2 - T_a}{R_{t4}} \quad ; \quad T_2 = Q_B R_{t4} + T_a$$

$$= 54.8 \times 0.987 + 27 = 81.09°C$$

**Example 3.47 The hot combustion gases at 300°C flow through a hollow cylindrical pipe of 10cm inner diameter and 12 cm outer diameter. The pipe is located in a space at 30°C and the thermal conductivity of the pipe material is 200 W/mK. Neglecting surface heat transfer coefficients, calculate the heat loss through the pipe per unit length and the temperature at a point halfway between the inner and outer surface. What should be the surface area normal to the direction of heat flow so that the heat transfer through the pipe can be determined by considering material of the pipe as a plane wall of the same thickness?**

**Solution:** In terms of geometrical parameters, thermal resistance of a pipe is

$$R_t = \frac{1}{2\pi kl} log_e r_2/r_1 = \frac{1}{2\pi \times 200 \times 1} log_e(12/10)$$

$$= 1.4516 \times 10^{-4} deg/W$$

Heat loss $Q = \frac{\Delta t}{R_t} = \frac{300-30}{1.4516 \times 10^{-4}} = 1860016.53\ W$

(ii) Radius at halfway through the pipe wall,

$$r = \frac{r_1 + r_2}{2} = \frac{10 + 12}{2} = 11cm$$

Thermal resistance of cylindrical pipe upto its mid-plane

$$= \frac{1}{2\pi kl} \log_e(r/r1) = \frac{1}{2\pi \times 200 \times 1} \log_e(11/10)$$

$$= 7.5884 \times 10^{-5} \text{deg/W}$$

Since heat flow through each section is same;

$$1860016.53 = \frac{t_1 - t}{7.5884 \times 10^{-5}}$$

Temperature at the mid plane,

$$t = 300\text{-}1860016.53 \times 7.5884 \times 10^{-5} = 158.85°C$$

Alternatively from the expression for temperature distribution

$$\frac{t - t_1}{t_2 - t_1} = \frac{\log_e(r/r_1)}{\log_e(r_2/r_1)}$$

$$t = t_1 - (t_1 - t_2) \frac{\log_e(r/r_1)}{\log_e(r_2/r_1)}$$

$$= 300 - (300 - 30) \frac{\log_e(11/10)}{\log_e(12/10)} = 158.85°C$$

(iii) The equivalent logarithmic mean area is

$$A_m = \frac{A_2 - A_1}{\log_e(A_2/A_1)}$$

$$= \frac{2\pi(r_2 - r_1)^l}{\log_e(r_2/r_1)} = \frac{2\pi\,(0.12 - 0.10) \times 1}{\log_e(12/10)} = 0.689m^2$$

Check:

$$Q = \frac{kA_m(t_1 - t_2)}{(r_2 - r_1)}$$

$$\frac{200 \times 0.689 \times (300 - 30)}{0.12 - 1.10} = W$$

This is approximately same as calculated above.

**Example 3.54 A steel pipe of 20 mm inner diameter and 2mm thickness is covered with 20mm thick of fibre glass insulation (k= 0.05 W/m-deg). If the inside and outside convective coefficients are 10 W/m²-deg and 5 W/m²-deg, calculate the overall heat transfer coefficient based on inside diameter of the pipe.**

**Solution:** $r_1$=10mm ; $r_2$ = 10+2=12mm ; $r_3$ = 12+20=32mm

The thermal resistances to flow of heat are offered by

(i) Inside fluid film, $R_{t_1} = \frac{1}{h_i A_i} = \frac{1}{h_i 2\pi r_1 l}$

(ii) Pipe material, $R_{t_2} = \frac{\log_e r_2/r_1}{2\pi k_1 l}$

(iii) Insulation, $R_{t_3} = \frac{\log_e r_3/r_2}{2\pi k_2 l}$

(iv) Outside air film, $R_{t_4} = \frac{1}{h_0 A_0} = \frac{1}{h_0 2\pi r_3 l}$

The heat transfer through the insulated pipe is than given by

$$Q = \frac{\Delta t}{\sum Rt}$$

$$= \frac{2\pi l\,(t_i - t_0)}{\frac{1}{h_i r_i} + \frac{1}{k_1}\log_e\left({}^{r_2}/_{r_1}\right) + \frac{1}{k_2}\log_e\left({}^{r_3}/_{r_2}\right) + \frac{1}{h_0 r_3}}$$

The thermal conductivity of steel pipe is not given, and generally it is much higher than that of insulation. Accordingly thermal resistance due to pipe material can be neglected.

That gives:

$$Q = \frac{2\pi l(t_i - t_0)}{\frac{1}{h_i r_i} + \frac{1}{k_2}\log_e\left({}^{r_3}/{}_{r_2}\right) + \frac{1}{h_0 r_3}} \qquad \text{.......(i)}$$

If $U_i$ is overall heat transfer coefficient based on inside area of the steel pipe, then heat flow rate can also be written as

$$Q = U_i A_i (t_i - t_0)$$

$$= U_i (2\pi r_1 l)(t_i - t_0) \qquad \text{.......(ii)}$$

Comparing identities (i) and (ii), we note that

$$U_i (2\pi r_1 l)(t_i - t_0) = \frac{2\pi l(t_i - t_0)}{\frac{1}{h_i r_i} + \frac{1}{k_2}\log_e\left({}^{r_3}/{}_{r_2}\right) + \frac{1}{h_0 r_3}}$$

Or $$\frac{1}{U_i r_1} = \frac{1}{h_i r_i} + \frac{1}{k_2}\log_e\left({}^{r_3}/{}_{r_2}\right) + \frac{1}{h_0 r_3}$$

Or $$\frac{1}{U_i} = \frac{1}{h_i} + \frac{r_1}{k_2}\log_e\left({}^{r_3}/{}_{r_2}\right) + \left({}^{r_3}/{}_{r_2}\right) \times \frac{1}{h_0}$$

Upon substitution of given data,

$$\frac{1}{U_i} = \frac{1}{10} + \frac{10\times10^{-3}}{0.05}\log_e(32/12) + (10/32)\times\frac{1}{5}$$

$$=0.1+0.196+0.0625=0.3585$$

$$U_i = \frac{1}{0.3585} = 2.789\ w/m^2 - deg$$

**Example 3.58 A 3 cm diameter pipe at 100°C is losing heat at the rate of 100 W per metre length of pipe to the surrounding air at 10°C. This is to be reduced to a minimum value by providing insulation. The following insulation materials are available:**

**Insulation A**

**Quantity =3.15per metre length of pipe**

**Thermal conductivity = 5W/m-deg**

**Insulation B**

**Quantity =4per metre length of pipe**

**Thermal conductivity = 1 W/m-deg**

**Examine the position of better insulating layer relative to the pipe. What percentage saving in heat dissipation results from that arrangement?**

**Solution:** Thermal resistance due to pipe material works out as

$$R_t = \frac{\Delta t}{Q} = \frac{100-10}{100} = 0.9\ deg/W$$

For a pipe with two layers of insulation,

$$\sum R_t = R_t + R_{t_1} + R_{t_2}$$

$$=0.9+\frac{1}{2\pi k_1 l}\log_e\frac{r_2}{r_1}+\frac{1}{2\pi k_2 l}+\frac{r_3}{r_2}$$

**1st Arrangement** : The insulation material A is placed inside, i.e., next to the pipe

$$\frac{\pi}{4}(r_2^2-r_1^2)l = 3.15\times10^{-3}$$

Or $\frac{\pi}{4}(r_2^2-0.015^2)\times1 = 3.15\times10^{-3}$ ; $r_2 = 0.065$ m

Also $\frac{\pi}{4}(r_3^2-r_2^2)l = 4\times10^{-3}$

Or $\frac{\pi}{4}(r_3^2-0.065^2)\times1 = 4\times10^{-3}$ ; $r_3 = 0.0965$ m

$$\therefore \quad \sum R_t = 0.9+\frac{1}{2\pi\times5\times1}\log_e\frac{0.065}{0.015}+\frac{1}{2\pi\times1\times1}\log_e\frac{0.0965}{0.065}$$

$$=0.9+0.0467+0.629=1.0096\ deg/W$$

$$\text{Heat loss, } Q = \frac{\Delta t}{\sum Rt} = \frac{100-10}{1.0096} = 89.14\text{W}$$

**2nd Arrangement** : The insulation material B is placed inside, i.e., next to the pipe

$$\frac{\pi}{4}(r_2^2 - r_1^2)l = 4 \times 10^{-3} \quad ; \quad r_2 = 0.073 \text{ m}$$

$$\frac{\pi}{4}(r_3^2 - r_2^2)l = 3.15 \times 10^{-3} \quad ; \quad r_3 = 0.0965$$

$$\sum R_t = 0.9 + \frac{1}{2\pi \times 1 \times 1} log_e \frac{0.073}{0.015} + \frac{1}{2\pi \times 5 \times 1} log_e \frac{0.0965}{0.073}$$

$$= 0.9 + 0.252 + 0.0089 = 1.1609 \text{ deg/W}$$

$$\text{Heat loss, } Q = \frac{100 - 10}{1.1609} = 77.526\text{W}$$

Obviously the heat loss is small when the insulation material B is placed next to pipe.

Saving in the heat loss

$$= \frac{100 - 77.526}{100} \times 100 = 22.47\%$$

**Example 3.63 Two insulation materials A and B, in powder form, with thermal conductivities of 0.005 W/m-deg and 0.03 W/m-deg were purchased for use over a sphere of 50 cm diameter. Material A was to form the first layer 4 cm thick and material B was to be the next layer 5 cm thick. Due to ovetsight during installation, whole of material B was applied first and subsequently there was a layer formed by material A. Investigate how the conduction heat transfer would be affected.**

**Solution:** **Case I** $r_1$=0.2 m ; $r_2$ = 0.24 m ; $r_3$ = 0.29 m

Thermal resistance to heat flow ,

$$R_{t_1} = \frac{r_2 - r_1}{4\pi k_1 r_1 r_2} + \frac{r_3 - r_2}{4\pi k_2 r_2 r_3}$$

$$= \frac{0.24 - 0.2}{4\pi \times 0.005 \times 0.2 \times 0.24} + \frac{0.29 - 0.24}{4\pi \times 0.003 \times 0.24 \times 0.29}$$

=13.27+1.906=**15.176°C/W**

**Case II :** When the materials get interchanged, there would be change in radii also.

$$\text{Volume of material A} = \frac{4\pi}{3}(0.24^3 - 0.2^3) = 0.024396 m^3$$

$$\text{Volume of material B} = \frac{4\pi}{3}(0.29^3 - 0.24^3) = 0.044255 m^3$$

The new radii are then worked out as

$$0.044255 = \frac{4}{3}\pi(r_2^3 - 0.2^3) \quad ; \quad r_2 = 0.2648\, m$$

$$0.024396 = \frac{4}{3}\pi(r_3^3 - 0.2648^3) \quad ; \quad r_3 = 0.29\, m$$

$$\text{Thermal resistance } R_{t2} = \frac{r_2 - r_1}{4\pi k_2 r_1 r_2} + \frac{r_3 - r_2}{4\pi k_1 r_2 r_3}$$

$$= \frac{0.2648 - 0.2}{4\pi \times 0.03 \times 0.2 \times 0.2648} + \frac{0.29 - 0.2648}{4\pi \times 0.005 \times 0.2648 \times 0.29}$$

$$=3.247+5.225=8.472°C/W$$

Heat transfer is inversely proportional to thermal resistance. As such the heat flow will increase by

$$\frac{15.176 - 8.472}{15.176} \times 100 = 44.17\%$$

**Example 3.65 A 6.5m diameter vertical kiln has a hemi-spherical dome at the top; the dome is fabricated from a 25 cm thick layer of chrome brick which has a thermal conductivity of 1.16 W/m-deg. The kiln dome has inside temperature of 875°C and 10°C atmospheric air result into 11.4 W/m$^2$-deg heat transfer coefficient between the dome and air. Estimate the outside surface temperature of the dome and the heat loss from kiln. Compare this heat with that would result from a flat dome fabricated from the same material and with kiln operating under identical temperature conditions.**

**Solution:** Conduction heat loss through from a spherical body is given by

$$= 4\pi k(t_1 - t_2) \times \frac{r_1 r_2}{r_2 - r_1}$$

And for a hemi-sphere it equals half of this value.

Conduction heat loss the hemi-spherical dome,

$$= \left[4\pi \times 1.16(875 - t_2) \times \frac{3.25 \times 3.5}{3.5 - 3.25}\right]$$

$$= 331.46(875 - t_2) \qquad \ldots(i)$$

Convective heat flow from outside surface of dome to the surrounding air,

$$= h\, A\, \Delta t$$

$$= 11.4 \times \left(\tfrac{1}{2} \times 4\pi \times 3.5^2\right) \times (t_2 - 10)$$

$$= 887(t_2 - 10) \qquad \ldots(ii)$$

Under Steady state conditions,

$$331.46(875 - t_2) = 887(t_2 - 10)$$

$$875 - t_2 = 2.65t_2 - 26.76$$

Temperature at the outside surface of the dome,

$$t_2 = \frac{875 + 26.76}{3.65} = 247.05°C$$

The heat loss from the dome may now be obtained from either of the expression (i) and (ii).

Q = 331.46(875-247.05) = 208140.307

(b) For a dome with flat top:

$$\frac{kA(t_1 - t_2)}{\delta} = hA(t_2 - t_a)$$

The area for conduction and convection heat flow will be same

$\therefore \qquad \frac{1.16(875-t_2)}{0.25} = 11.4(t_2 - 10)$

Or

$$875 - t_2 = 2.45t_2 - 24.569$$

Or $\qquad t_2 = \frac{875+24.569}{3.45} = 260.74°C$

$$Q = 11.4 \times (\pi \times 3.25^2) \times (260.74 - 10) = 94803.602W$$

$$\text{Reduction in heat loss} = \frac{208140.307 - 94803.602}{208140.307} = 0.544 \ or \ 54.4\%$$

**Example 4.1 The rear window of an automobile is made of 5 cm thick glass of thermal conductivity 0.8 W/m-deg. To defrost this window, a thin transparent film type heating element has been fixed to its inner surface. For the conditions given below, determine the electric power that must be provided per unit area of window if a temperature 5°C is maintained at its outer surface.**

**Interior air temperature and the corresponding surface coefficient,= 20°C and 12 W/m$^2$-deg. Surrounding air temperature and the corresponding surface coefficient,= -15°C and 70 W/m$^2$-deg.**

**Electric heater provides uniform heat flux.**

**Solution:**

Given: $t_i = 20°C$ ; $h_i = 12\ W/m^2 - deg$ ; $t_0 = 15°C$ ; $h_0 = 70\ W/m^2 - deg$ ; $t_s = 5°C$

For unit area, the heat balance provides

$$\frac{(t_i - t_s)}{\frac{1}{h_i} + \frac{\delta}{k}} + q_g = h_0(t_s - t_0)$$

Substituting the given data,

$$\frac{(20 - 5)}{\frac{1}{12} + \frac{0.005}{0.8}} + q_g \ = 70 \times [5 - (-15)]$$

Or $$\frac{15}{0.833 + 0.00625} + q_g = 1400$$

∴ Electric power to be provided,

$$q_g = 1400 - \frac{15}{0.08955}$$

x=1400-167.50 =**1232.5 W/m²**

**Example 4.2 A composite slab consists of 2 cmthick layer of steel (k=146 kj/m-hr-deg) on the left side and a 6 cm thick layer of brass (k = 276 kj/m-hr-deg) on the right hand side. The outer surfaces of the steel and brass layer are maintained at 100° and 50° respectively. The contact between the two slab is perfact and heat is generated at the rate of 4.2 $10^5$ kj/m²-hr at the plane of contact. The heat thus generated is dissipated from both sides of composite slab for steady state conditions. Calculate the temperature at the interface and heat flow through each slab.**

**Solution:** Let $t_i$ be the temperature at the interface. Under stipulation for heat dissipation from both sides,

$t_i > t_1 > t_2$

Accordingly we may write

$$Q_1 + Q_2 = Q_g$$

$$\frac{k_1 A_1 (t_i - t_1)}{\delta_1} + \frac{k_2 A_2 (t_i - t_2)}{\delta_2} = Q_g$$

Considering unit surface area

$$\frac{146 \times 1 \times (t_i - 100)}{0.05} + \frac{276 \times 1 \times (t_i - 50)}{0.06} = 4.2 \times 10^5$$

Or $\quad 2920(t_i - 100) + 4600(t_i - 50) = 4.2 \times 10^5$

Or $\quad 7520\ t_i = 4.2 \times 10^5 + 2.92 \times 10^5 + 2.3 \times 10^5$

$$= 9.42 \times 10^5$$

Temperature at the interface,

$$t_i = \frac{9.42 \times 10^5}{7520} = 125.26°C$$

Heat transfer through the steel layer,

$$Q_1 = \frac{146 \times 1 \times (125.26 - 100)}{0.05} = 7359 \text{ kj/m}^2\text{-hr}$$

Heat transfer through the brass layer,

$$Q_1 = \frac{276 \times 1 \times (125.26 - 50)}{0.06} = 346196 \text{ kj/m}^2\text{-hr}$$

**Example 4.7 A long stainless steel bar 20mm 20mm in square cross-section is perfectly insulated on three sides and is maintained at a temperature of 400°C on the remaining side. Determine the maximum temperature in the bar when it is conducting a current of 1000 ampere. Take thermal and electrical conductivities of steel as 16 W/m-deg and 1.5 /ohm-cm and neglect the edge effects. Also work out the heat flow the bar.**

**Solution:** The heat generated per unit volume due to flow electric current is worked out from the relation.

$$Qg = \left(\frac{1}{A}\right)^2 \rho$$

Where ρ is resistivity in ohm-cm, i.e. , reciprocal of electrical conductivity

$$Qg = \left(\frac{1000}{2 \times 2}\right)^2 \times \frac{1}{1.5 \times 10^4} = 4.167 W/cm^3 = 4.167 \times 10^6 W/m^3$$

The temperature distribution through the bar is prescribed by the relation

$$t = \left[\frac{q_g}{2k}(\delta - x) + \frac{(t_2 - t_1)}{\delta}\right] x + t_1$$

Maximum temperature occurs at the centre, i.e. , at Further under the given boundary conditions: Therefore,

$$t_{max} = \frac{q_g}{2k}\left(\delta - \frac{\delta}{2}\right)\frac{\delta}{2} + t_1$$

$$= \frac{q_g}{8k}\delta^2 + t_1$$

$$= \frac{4.167 \times 10^6}{8 \times 16}(0.02)^2 + 400 = 413.02°C$$

Under steady state conditions, the heat flow through the bar equals the heat generated within it

$$= q_g \times volume\ of\ the\ bar$$

$$= (4.167 \times 10^6) \times (0.02 \times 0.02 \times 1)$$

$$= 1.67 \times 10^3\ w/m\ length\ of\ the\ bar$$

**Example 4.21 A stainless steel wire (conductivity = 20 W/m-deg and resistivity=70 micro ohm-cm) of length 2 m and diameter 2.5 mm is submerged in a fluid at 50°C and an electric current of intensity 300 amps passes through it. If conductance at the wire surface is 4 kW/m²-deg, workout the steady state temperature at the centre and at the surface of the wire.**

**Solution:**

Electrical resistance of wire, $R_e = \frac{\rho l}{A}$

$$= \frac{70 \times 10^{-6} \times 200}{\frac{\pi}{4} \times (0.25)^2} = 0.285\Omega$$

Heat generated, $Q_g = I^2 R_e = 300^2 \times 0.285$ watt

Volume of wire, $V = \frac{\pi}{4} d^2 l = \frac{\pi}{4}\left(\frac{2.5}{1000}\right)^2 \times 2 = 9.81 \times 10^{-6} m^3$

Heat generated per unit volume, $q_g = \frac{300 \times 0.285}{9.81 \times 10^{-6}} = 2.615 \times 10^9\ W/m^3$

Radius of wire, $R = \frac{2.5/2}{1000} = 0.00125\text{m}$

The wire surface temperature is given by,

$$t_w = t_a + \frac{q_g}{2h} R$$

$$t_w = 50 + \frac{2.615 \times 10^9}{2 \times (4 \times 1000)} \times 0.00125$$

$$= 50 + 408.59 = 458.59°\text{C}$$

Maximum temperature in the wire occurs at its geometric centre line, and can be computer from the relation,

$$t_{max} = t_a + \frac{q_g}{2h} R + \frac{q_g}{2k} R^2$$

$$= t_w + \frac{q_g}{2k} R^2$$

$$= 458.59 + \frac{2.615 \times 10^9}{4 \times 20} \times (0.00125)^2$$

$$= 458.59 + 51.07 = 510.66°\text{C}$$

**Example 4.24 A 66 V transmission line carrying a current of 850 ampere is 20 mm in diameter and electrical resistance of the copper conductor is 0.075 ohm/km. Assuming that the surrounding are at 38°C and that the combined convection and radiation coefficient for**

**heat transfer from the wire surface to the surroundings is 14.2 W/m² K, make calculations for:**

**(i) Surface temperature of the transmission line**

**(ii) Rate of heat generation per unit volume of the wire.**

**(iii) Maximum temperature in the line.**

**The thermal conductivity of copper is 400 W/mK.**

**Solution**

(i) Heat generated in the transmission line due to flow of current

$$= I^2 R$$

$$= 850^2 \times 0.0750 W/km$$

$$= \frac{850^2 \times 0.075}{1000} = 54.187 \ W \ per \ metre \ length$$

Heat dissipated to surroundings by combined convection and radiation

$$= h \ A \ \Delta t$$

$$= 14.2(\pi \times 0.02 \times 1)(t_w - 38) \ W \ per \ metre \ length$$

Under steady state conditions

$$54.187 = 14.2(\pi \times 0.02 \times 1)(t_w - 38)$$

Solving, $t_w$ (wire surface temperature) =98076°C

(ii) Let $q_g$ be the volumetric heat generated at uniform rate over the wire cross-section

$$54.187 = q_g \times \left\{\pi \times \left(\frac{0.02}{0}\right)^2 \times 1\right\} = 3.14 \times 10^{-4} q_g$$

$$\therefore \ q_g = \frac{54.187}{3.14 \times 10^{-4}} = 1.726 \times 10^5 W/m^3$$

(iii) Maximum temperature in the wire will occur at the geometric centre line of the wire and may be computed from the relation,

$$t_{max} = t_w + \frac{q_g}{4k} R^2$$

$$= 98.76 + \frac{1.726 \times 10^5}{4 \times 400} (0.01)^2$$

$$= 98.76 + 0.01079 = 98.77079°C$$

The small difference between surface and centre temperature results from the relatively small heat generation rate and the high thermal conductivity of copper.

**Example 4.26 An internally copper conductor of 2 cm outer radius and 0.75 cm inner radius carries a current density of 5000amp/cm². A constant temperature of 70° C is maintained at the inner surface and there is no heat transfer through insulation surrounding the copper. Set up an equation for temperature distribution through copper. Proceed to calculate the maximum temperature of copper and the radius at which it occurs. Also find the internal heat transfer rate and check that this equals the total energy generation in the conductor.**

**For copper: thermal conductivity k= 380 W/m-deg and the resistivity**

**Solution** Total volumetric heat generation,

$$= I^2 R = I^2 \frac{\rho l}{A}$$

Heat generated per unit volume,

$$q_g = I^2 \frac{\rho l}{A} \div Al = \rho \left(\frac{I}{A}\right)^2$$

$$= 2 \times 10^{-8} (5000 \times 10^4)^2 = 50 \times 10^6 W/m^3$$

For steady state conditions, the radial temperature distribution for a hollow cylinder with outside surface insulated is given by

$$t = t_1 + \frac{q_g}{2k}\left[\frac{r_1^2 - r^2}{2} + r_2^2 \log_e\left(\frac{r}{r_1}\right)\right]$$

The maximum temperature occurs at the insulated surface, i.e. ,at the outer radius and it equals

$$t_{max} = t_1 + \frac{q_g}{2k}\left[\frac{r_1^2 - r^2}{2} + r_2^2 \log_e\left(\frac{r_2}{r_1}\right)\right]$$

Inserting the appropriate values,

$$t_{max} = 70 + \frac{50 \times 10^6}{2 \times 380}\left[\frac{0.0075^2 - 0.02^2}{2} + 0.02^2 \log_e\left(\frac{0.02^2}{0.0075}\right)\right]$$

$$=70+65789.5[-0.0001719+0.0003923]$$

$$=70+14.50=84.5°C$$

The internal heat transfer rate can be obtained by finding the temperature gradient at the inner radius, i.e., at r = 0.0075m and then invoking the Fourier's law of heat conduction.

$$t = \frac{q_g}{4k}(-2r) + \frac{q_g}{2k} r_2^2 \left(\frac{1}{r}\right) = \frac{q_g}{2k} r + \frac{q_g}{2k} \frac{r_2^2}{r}$$

$$\left|\frac{dt}{dr}\right|_{r=r_1} = -\frac{q_g}{2k} r_1 + \frac{q_g}{2k} \frac{r_2^2}{r}$$

$$= -\frac{50 \times 10^6}{2 \times 380} \times 0.0075 + \frac{50 \times 10^6}{2 \times 380} \times \frac{0.02^2}{0.0075}$$

$$= -493.42 + 3508.77 = 3015.35$$

$$\therefore Q = -kA\frac{dt}{dr} = -380 \times (2\pi \times 0.0075 \times 1) \times 30.15.35$$

$$= -53968.7 \text{ W/m length of conductor}$$

The –ve sign indicates that the heat flow is radially inwards.

**Check:** Since the outer surface is insulated, the entire heat generated within the conductor must be dissipated internally. Therefore the internal heat transfer must be

$$= (\text{volume per m length of conductor}) \times q_g$$

$$= \pi(0.002^2 - 0.0075^2) \times 50 \times q_g$$

$$= 53968 \frac{W}{m} \, lengthof \; conductor$$

**Example 4.29 A hollow cylinder of 3 cm inner radius and 4.5 cm outer radius has a heat generation rate of . The inner and outer surfaces are maintained at temperatures of 380°C and 360°C respectively and thermal conductivity of the cylinder material is 20 W/m-deg. Make calculations for the temperature at mid radius.**

**Solution** $r_1$=0.03 m ; $r_2$ = 0.45 m ; and $r_3$ = 0.0375 m at mid radius

For the specified boundary conditions, the temperature distribution is given by

$$t - t_1 = \frac{q_g}{2k}(r_1^2 - r^2) + \left[(t_1 - t_2) - \frac{q_g}{4k}(r_2^2 - r_1^2)\right]\frac{\log_e(r/r_1)}{\log_e(r_1/r_2)}$$

Inserting the appropriate data,

$$t - 380 = \frac{5\times10^6}{4\times380}(0.03^2 - 0.0375^2)$$

$$+\left[(380 - 360) - \frac{5\times10^6}{4\times30}(0.045^2 - 0.03^2)\right]\frac{\log_e(0.0375/0.03)}{\log_e(0.03/0.045)}$$

$$= -21.09 + (20 - 46.875)\times\frac{0.223}{(-0.4055)} = -6.31$$

∴ Temperature at mid radius, t=380-6.31=373.69°C

# Lesson 7. Numericals on conduction

## Thermal Insulation

The purpose of use of thermal insulation is to prevent or reduce transfer of thermal energy between two systems maintained at different temperatures. Thermal insulation is generally used in following applications

- Protective clothing for human comfort
- Design of energy efficient buildings
- Air Conditioning systems
- Refrigeration and food preservation
- Automobiles
- Boilers and steam pipes
- Spacecraft

Value of thermal conductivity of a material is generally used as a measure of its insulating capabilities. Materials having lower value of thermal conductivity are considered to be insulators. Insulating capability of a material depends upon following factors

- Thermal conductivity
- Temperature
- Density or Porosity
- Specific heat
- Surface emissivity
- Moisture Content
- Air Pressure
- Convection with in insulating material

According to 'Thermal Insulation Association of Canada' thermal insulation is used for temperature range of -75 $^{o}$C and 815 $^{o}$C as for temperature range below -75 $^{o}$C and higher than 815 $^{o}$C, cryogenic and refractory materials are used respectively. Depending upon temperature range for which thermal insulation is to be used, insulating materials are categorized as

i) Low temperature insulating materials for temperature range of -75 $^{o}$C to 15 $^{o}$C

ii) Medium temperature insulating materials for temperature range of 15 $^{o}$C to 315 $^{o}$C

iii) High temperature insulting materials for temperature range of 315 °C to 815 °C

The most commonly used insulating materials and their properties are given in Table 1

**TABLE 1– Thermal Conductivities of Insulating Materials (Powders)**

| | **Material** | **Mean Temperature °C** | **Conductivity (K) k cal/ m-hr-°C** |
|---|---|---|---|
| **1.** | Alumina compressed powder | 50 | 7.0 |
| **2.** | Ashes, Soft wood | 25 | 0.33 |
| **3.** | Carbon black | 55 | 0.22 |
| **4.** | Coal dust | 40<br>100 | 1.20<br>1.32 |
| **5.** | Coke dust | 25 | 1.50 |
| **6.** | Charcoal | 15 | 0.54 |
| **7.** | Cork-granulated | 0 | 0 45 |
| **8.** | Floto foam | 35 | 0.30 |
| **9.** | Graphite powder | 40 | 1.40 |
| **10.** | Plaster of Paris | 25 | 10.5 |
| **11.** | Fine river sand Moistured | 10 | 3.3 |
| **12.** | Saw dust | 25 | 11.6 |
| **13.** | Silica | 30 | 0.06 |

| | | | |
|---|---|---|---|
| **14.** | Silica Aerogel | 315<br>480<br>650 | 0.80<br>0.84<br>0.93 |
| **15.** | Silica gel | -70<br>-20<br>40<br>100<br>150<br>55 | 0.174<br>0.207<br>0.240<br>0.276<br>0.318<br>0.90 |
| **16.** | Snow | 0 | 0.51 to 1.9 |

**Critical Thickness of Insulation**

Addition of layer insulation to walls and slabs always reduces the heat transfer, however, in case of systems having cylindrical and spherical surfaces, addition of insulation does not reduce the heat transfer, rather it increases heat transfer rate upto certain thickness of insulation. As insulation layer is added to a bare pipe, heat transfer rate increases instead of decreasing. With further addition insulation layer, heat transfer rate goes on increasing till it attains a maximum value beyond which it decreases with increase in thickness of insulation material. The thickness of insulation corresponding to maximum heat transfer rate from the pipe is called critical thickness of insulation.

This concept of critical thickness of insulation finds a useful application in electric transmission cables. Transmission losses increase with increase in temperature as electric current passes through the cables. In order to reduce the transmission losses, it becomes imperative to increase heat transfer rate from cables to surroundings so that temperature of cables is reduced. This is achieved by keeping the thickness of insulation equal to its critical thickness.

In order to determine the critical thickness of insulation, consider a cylinder of negligible thickness, length 'L', radius '$r_1$' carrying hot fluid of temperature $T_1$. Temperature of hot fluid is higher than that of ambient temperature $T_a$. The cylinder is insulated by an insulating material having thickness't' and thermal conductivity 'k'. Thickness of insulation can be expressed as

$$t = r2 - r1 \qquad (1)$$

where $r_2$ is the outer radius of the arrangement consisting of thin cylinder and layer of insulating material and it depends upon value of the thickness of insulating material. The arrangement has been shown in Figure 1.

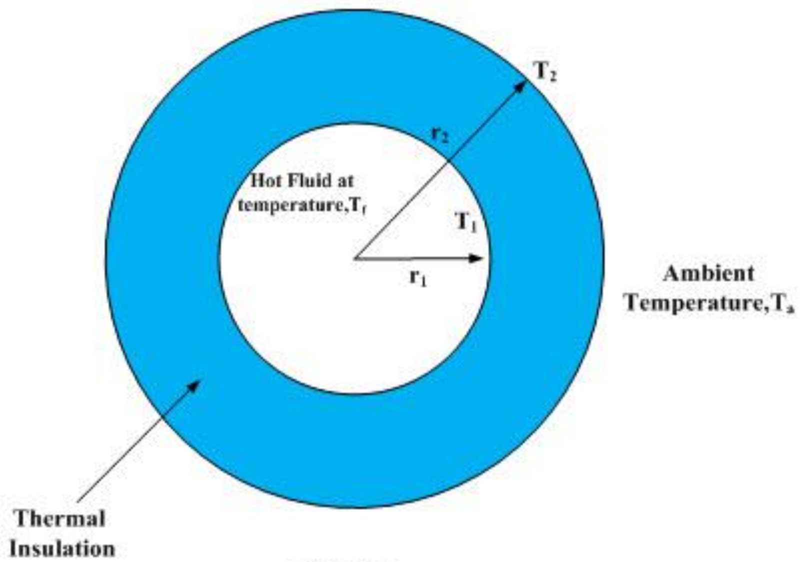



Figure 1

Heat transfer rate per unit length from cylinder is expressed as

$$Q = \frac{T_1 - T_a}{\frac{1}{2\pi r_1 L}\frac{1}{h_i} + \frac{1}{2\pi k L}\log_e\left(\frac{r_2}{r_1}\right) + \frac{1}{2\pi r_2 L}\frac{1}{h_o}} \qquad (2)$$

$h_i$ and $h_o$ inside and outside heat transfer coefficients of convection

By using equation (1), heat transfer rate per unit length can be determined for different value of $r_2$. Total thermal resistance of the arrangement is sum of inside convective thermal resistance, conductive thermal resistance of insulating material and outside convective thermal resistance. Total thermal resistance of the arrangement depends upon the value of $r_2$ which in turn depends upon the thickness of insulating material. As the thickness of insulation increase, $r_2$ also increase resulting in increase in conductive thermal resistance and decrease in outer convective resistance. Therefore, increase in thickness of insulation will either result in increase or decrease in heat transfer rate depending on the overall change in the total thermal resistance.

For heat transfer rate per unit length to be maximum, thermal resistance should be the minimum. The value of $r_2$, for which heat transfer rate per unit length will be maximum, can be obtained by differentiating total thermal resistance i.e. denominator of equation (2) with respect to $r_2$ and equating equal to zero.

$$\frac{d}{dr_2}\left(\frac{1}{2\pi r_1}\frac{1}{h_i}+\frac{1}{2\pi k}\log_e\left(\frac{r_2}{r_1}\right)+\frac{1}{2\pi r_2}\frac{1}{h_o}\right)=0$$

$$\frac{1}{2\pi k}\frac{1}{r_2}-\frac{1}{2\pi h_o}\frac{1}{r_2^2}=0$$

$$\frac{1}{2\pi r_2}\left(\frac{1}{k}-\frac{1}{r_2 h_o}\right)=0$$

$$r_2=\frac{k}{h_o}=r_c \qquad (3)$$

Equation (3) gives the value of thickness of insulation for which heat transfer rate per unit length is the maximum as corresponding total thermal resistance is the minimum. The value of thickness of insulation at which total thermal resistance is minimum is called critical thickness of insulation and is represented by $r_c$ and depends upon thermal conductivity and convective heat transfer coefficient of insulating material. Addition of insulation to a bare surface will either increase or decrease the heat transfer rate per unit lenth depending upon the value of critical radius and radius of bare surface.

i) If $r_1 < r_c$, the rate of heat transfer per unit length of cylinder increases as thickness of insulation increases. The heat transfer rate goes on increasing with increase in thickness of insulation and attains a maximum value corresponding to that value of insulation of thickness for which $r_2$ becomes equal to $r_c$ as shown in Figure 2 . It is due to the fact that in ranger $r_1$ less than $r_c$, progressive decreases in convective resistance with increase in insulation thickness dominates over the corresponding increase in conductive resistance. The net effect is decreases in total thermal resistance and heat transfer rate per unit length increases.

ii) For the range $r_1$ greater $r_c$, heat transfer rate goes on decreasing with increase in insulation thickness as shown in Figure 3. It is attributed to the fact that in range $r_1$ greater than $r_c$, effect of conductive resistance dominates over that of the convective resistance resulting in increases in total thermal resistance. Therefore, heat transfer rate per unit length decreases.

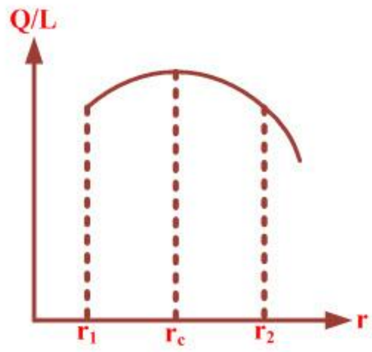



Figure 2

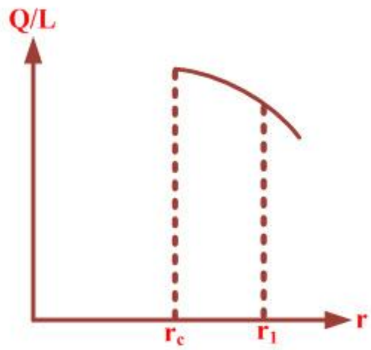



Figure 3

**Lesson 8. Insulation materials, critical thickness of insulation and Numerical Problems**

HEAT TRANSFER FROM EXTENDED SURFACES

**Fins:** Heat transfer from a hot surface to surrounding fluid takes place by convection and is governed by Newton's law of cooling

$$Q = h\,A\,(T_s - T_f) \qquad (1)$$

It is obvious from above equation that heat transfer rate can be enhanced by increasing

- Either by increasing convective heat transfer coefficient, h.
- Or by increasing area of heat transferring surface, A.
- Or by increasing temperature difference, $(T_s - T_f)$

Heat transfer rate is generally enhanced by increasing area of heat transferring surface as shown in Figure 1. It is not possible always to increase the value of convective heat transfer coefficient, h, by increasing velocity of flow of fluid surrounding the hot surface or to increase temperature difference by lowering temperature of fluid which is in contact with the hot surface.

Increase in heat transfer area is achieved by attaching protrusions to hot solid surface. These protrusions are referred as fins or spines and are made of materials having high thermal conductivity such as aluminum. Fins bring about considerable enhancement in heat transfer rate from a solid surface by exposing a larger surface area for convective and radiative heat transfer.

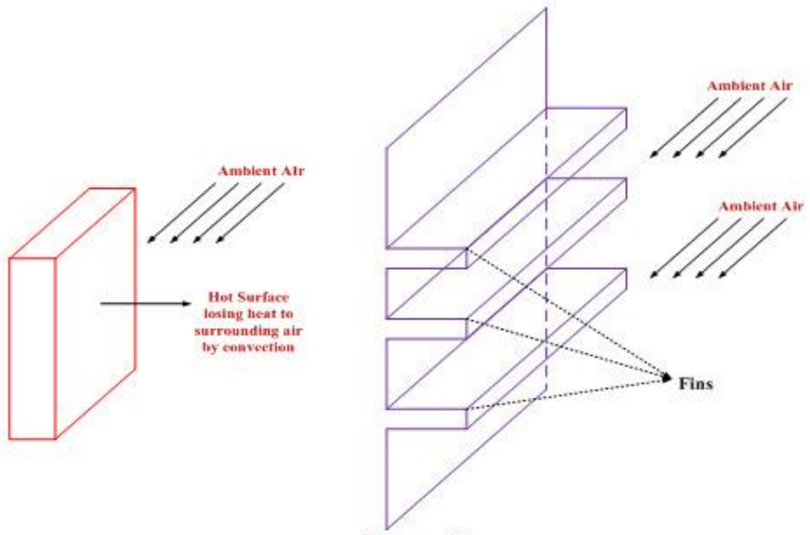



**Figure 1**

**Fin Applications:-** Fins are generally used when convective heat transfer coefficient is low such as in case of gases and under natural convection. There are numerous appliances used by us in daily life in which fins have been used to enhance heat transfer rate.

- Electrical apparatus like transformers and motors
- Engines of scooters, motorcycles and compressors The circumferential fins of rectangular or triangular profile are commonly used on the engine cylinders of scooters and motor cycles
- Condenser and evaporators of a refrigerating systems
- Car radiator

**Types of Fins:**

1. Straight Fin: The terms straight fin is applied to the extended surface attached to a wall which is otherwise plane. Figures 2 (a) and (b) are of fins of uniform area whereas Figures (c) and (d) are of fins of non-uniform area.

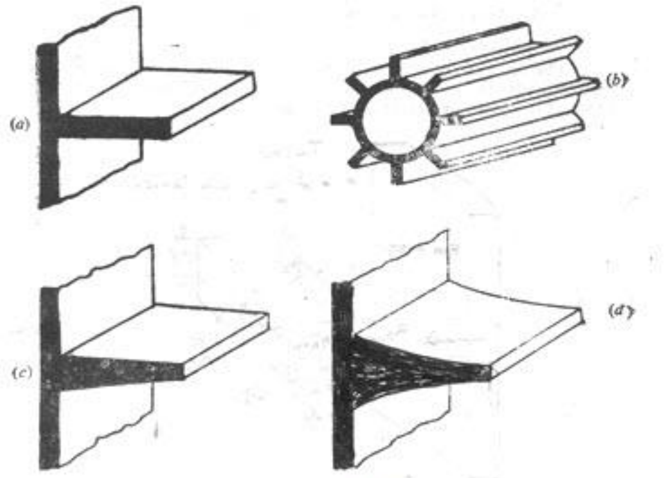



**Figure 2**

2. Annular(circumferential) fin: Annular fin is one, attached circumferentially, to a cylindrical surface as showon in Figure 3.

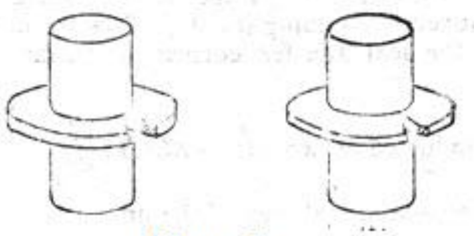


**Figure 3**

3. **Spine or pin fin:** It is an extended surface of cylindrical or conical shape as shown in Figure 4.

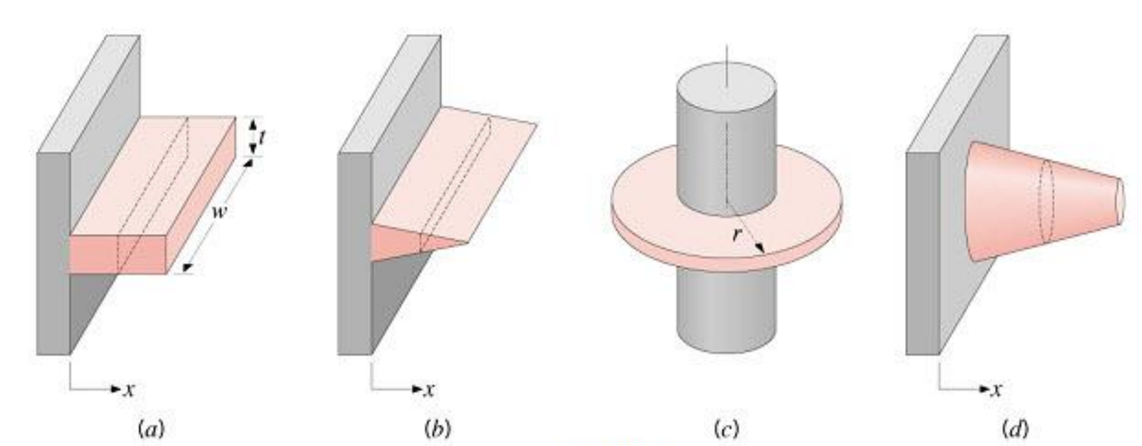



Figure 4

**Heat transfer through a fin of uniform cross-section**

The purpose of heat transfer analysis trough a fin is to determine the increases in heat transfer obtained by attaching fins to a surface. In order to increase heat transfer from a plane wall, a rectangular fin of length 'L', width 'w' and thickness '2δ' has been attached to the wall as shown in Figure 5. Heat flows along the length of the fin and it is surrounded by atmospheric air at temperature $T_a$. Temperature at the base or root of fin adjoining the wall is $T_o$.

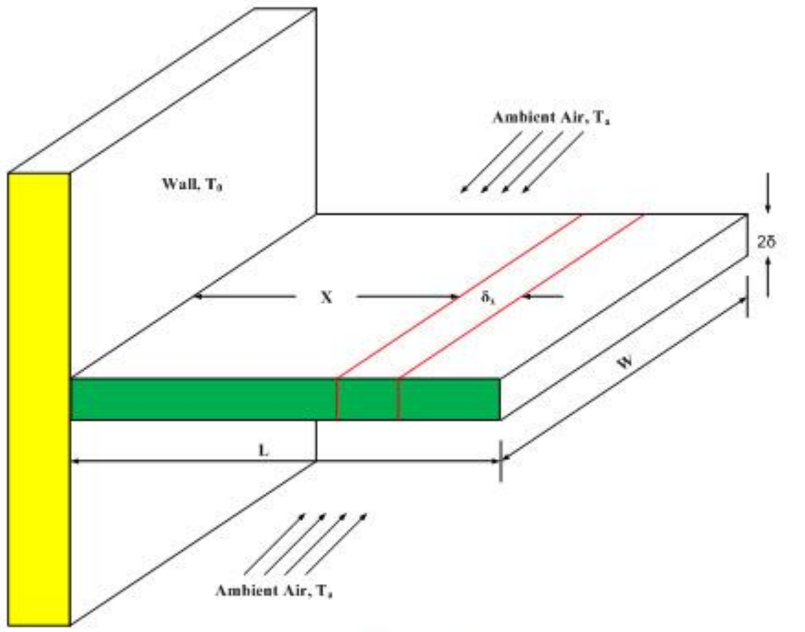



Figure 5

In order to simplify the analysis, following assumptions have been made

- Flow of heat takes place under steady state conditions and is one dimensional i.e. along the length of fin.
- Thermal conductivity of fin material is constant.
- Convective heat transfer coefficient 'h' is uniform over the entire surface of fin.
- Temperature, $T_o$, at the base or root of fin is uniform and is equal to wall temperature $T_w$.

Consider a small element of thickness '$\delta_x$' at a distance 'x' from the base of fin and its perimeter and cross-sectional area are expressed as

Perimeter, $P = 2w + 4\delta = 2(w + 2\delta)$

Cross-sectional area, $A_c = w$

Surface Area, $A = P\,\delta_x$

Under steady state conditions, energy balance on this element can be expressed as

| | | | | |
|---|---|---|---|---|
| Rate of heat conducted in to the element at x | = | Rate of heat conducted from the element at x+ $\delta_x$ | = | Rate of heat convected from the element |

Heat conducted into the element = $Q_x = -k A_c \frac{dT}{dx}|_{x=x}$ (2)

Heat conducted out of the element = $Q_{x+\delta x} = Q_x + \frac{d}{dx}(Q_x)\,\delta_x = -k A_c \frac{dT}{dx}|_{x=x+\delta x}$ (3)

Heat convected out of the element = $Q_c = h\,A\,(T - T_a) = h\,\delta_x\,P\,(T - T_a)$ (4)

Heat balance for the element is given by-

$$Q_x = Q_{x+\delta x} + Q_c \quad (5)$$

Using equations ( 2), (3), (4), equation (5) can be written as

$$-k A_c \frac{dT}{dx}|_{x=x} = -k A_c \frac{dT}{dx}|_{x=x+\delta x} + h\,\partial x P(T - T_a) \quad (6)$$

$$-k A_c \frac{dT}{dx}|_{x=x} = -k A_c \frac{dT}{dx}|_{x=x} + \frac{d}{dx}(Q_x)\,\delta_x + h\,\delta_x\,P(T - T_a)$$

$$\frac{d}{dx}\left(-k A_c \cdot \frac{dT}{dx} \cdot \delta_x\right) = -\,h\,\delta_x\,P(T - T_a)$$

$$-kA_c\,\delta x\,\frac{d^2T}{dx^2} = -h\,\delta x\,P(T-T_a)$$

$$kA_c\frac{d^2T}{dx^2} = hP(T-T_a) \qquad (7)$$

Let $T-T_a=\theta$ (8)

and $T_o-T_a=\theta_o$ (9)

Differentiating equation (8) with respect to x twice, we get

$$\frac{dT}{dx}=\frac{d\theta}{dx}, \Rightarrow \frac{d^2T}{dx^2} = \frac{d^2\theta}{dx^2} \qquad (10)$$

Using equation (10), equation (7) can be written as

$$kA_c\frac{d^2\theta}{dx^2} = hP\theta$$

$$\Rightarrow \frac{d^2\theta}{dx^2} = \frac{hP}{kA_c}\theta \quad \Rightarrow \quad \frac{d^2\theta}{dx^2} = m^2\theta$$

where $m^2=\frac{hP}{kA_c}$ or $m=\sqrt{\frac{hP}{kA_c}}$

$$\Rightarrow \frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad (11)$$

Equation (11) is the governing equation and its solution is expressed as

$\theta = Ae^{mx}+Be^{-mx}$

$\because e^{mx}=\cosh mx + \sinh mx;\ e^{-mx}=\cosh mx - \sinh mx$

$\therefore \theta = (A+B)\cosh mx+(A-B)\sinh mx$

$\Rightarrow \theta = c_1\cosh mx + c_2\sinh mx$ (12)

Where $C_1$ = A+B and $C_2$ = A-B

Equation (12) represents general

**Lesson 9. Types of Fins, Fin Applications, Heat Transfer through Fin of uniform cross-section**

**Special cases**

General governing equation for heat transfer from a finned surface is expressed as

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad (1)$$

Soultion of equation (1) is expressed as

$$\theta = c_1 \cosh mx + c_2 \sinh mx \qquad (2)$$

Using the above two equations, following special cases are considered for heat transfer through a fin of uniform cross section:

**1. Fin is losing heat at the tip only**

When a fin of finite length loses heat only at its tip as shown in Figure 1, the relevant boundary conditions are

i) **At the base of fin, its temperature is equal to the wall temperature. Therefore**,

At $x = 0$; $T = T_0$

$T - T_a = T_0 - T_a$

or $\theta = \theta_0$ (3)

$\theta_0$ represents temperature difference between the fin base and the fluid surrounding the fin.

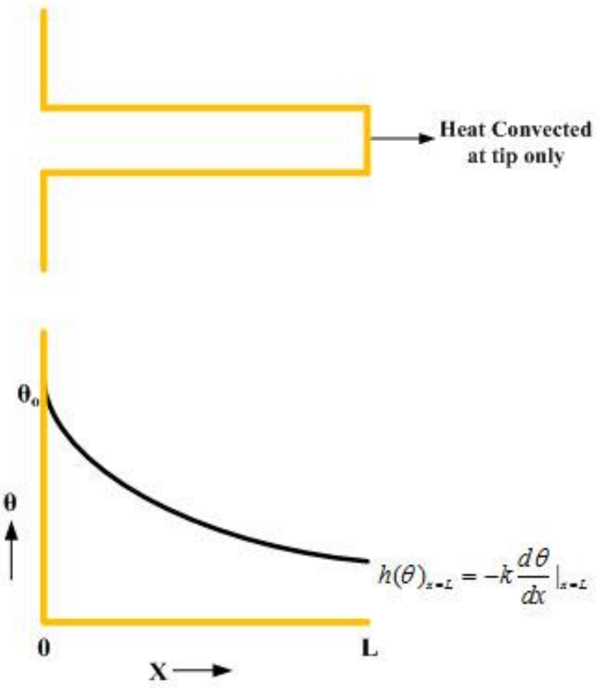



Figure 1

Applying first boundary condition to equation (2), we get

$\theta_o = c_1 \cosh m \times 0 + c_2 \sinh m \times 0$

$C_1 = \theta_o$ (4)

**ii) As fin is losing heat at tip only, it means heat conducted through fin is lost to surrounding fluid by convection at tip. Therefore,**

At $x = L$;

Heat conducted through fin = Heat convected to surrounding fluid by convection

$$-kA_c\frac{dT}{dx}|_{x=L} = hA_c(T_{x=L} - T_a)$$

$$-k\frac{d\theta}{dx}|_{x=L} = h(\theta)_{x=L} \qquad (5)$$

Substituting the value of $C_1$ from equation (4) in equation (2), we get

$\theta = \theta_o \cosh mx + c_2 \sinh mx$ (6)

Differentiating equation (6) with respect to x, we get

$$\frac{d\theta}{dx} = \theta_o m \sinh mx + mC_2 \cosh mx \qquad (7)$$

Substituting value of   from equation (6) in equation (5), we get

$$-k(\theta_o m \sinh mx + mC_2 \cosh mx)_{x=L} = h(\theta)_{x=L}$$

$$-k(\theta_0 m \sinh mL + c_2 m \cosh mL) = h(\theta_0 \cosh mL + c_2 \sinh mL)$$

Solving above equation for value of $C_2$, we can write

$$c_2 = \frac{-\theta_0 (mk \sinh mL + h \cosh mL)}{(h \sinh mL + mk \cosh mL)} \tag{8}$$

Substituting the values of $C_1$ and $C_2$ in equation (6) we get

$$\theta = \theta_0 \cosh mx - \theta_0 \frac{(mk \sinh mL + h \cosh mL)}{(h \sinh mL + mk \cosh mL)} \sinh mx \tag{9}$$

Equation (9) represents temperature distribution.

Rate of heat transfer from fin is expressed as

$$Q = -kA_c \frac{dT}{dx}\Big|_{x=0} = -kA_c \frac{d\theta}{dx}\Big|_{x=0} \tag{10}$$

As we know that $\frac{dT}{dx} = \frac{d\theta}{dx}$

Differentiating equation (9) with respect to x, we get

$$\left(\frac{d\theta}{dx}\right) = \theta_o m \sinh mx - \theta_o m \left(\frac{h \cosh mL + mk \sinh mL}{mk \cosh mL + h \sinh mL}\right) \cosh mx$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = 0 - \theta_o m \left(\frac{h \cosh mL + mk \sinh mL}{mk \cosh mL + h \sinh mL}\right) \tag{11}$$

Substituting the value of $\left(\frac{d\theta}{dx}\right)_{x=0}$ from above equation in equation (10), we get

$$\Rightarrow Q = kA_c\theta_0 m\left(\frac{mk\sinh mL + h\cosh mL}{h\sinh mL + mk\cosh mL}\right) \qquad (12)$$

Equation (12) can be written as

$$Q = kA_c\theta_0 m\left(\frac{\sinh mL + \frac{h}{mk}\cosh mL}{\cosh mL + \frac{h}{mk}\sinh mL}\right) \qquad (13)$$

Substituting the value of $m = \sqrt{\frac{hP}{kA_c}}$ and $\theta_0 = T_0 - T_a$ in above equation, we get

$$Q = \sqrt{PhkAc}\,(T_o - T_a)\left(\frac{\sinh mL + \frac{h}{mk}\cosh mL}{\cosh mL + \frac{h}{mk}\sinh mL}\right) \qquad (14)$$

Multiplying and dividing the left hand side of above equation by coshmL, we get

$$Q = \sqrt{PhkAc}\,(T_o - T_a)\left(\frac{\tanh mL + \frac{h}{mk}}{1 + \frac{h}{mk}\tanh mL}\right) \qquad (15)$$

The equation gives represents rate of heat transfer from a fin which is losing heat at the tip only.

**2. Fin is sufficiently long or infinite**

For a fin of infinite length as shown in Figure 2, the relevant boundary conditions are:

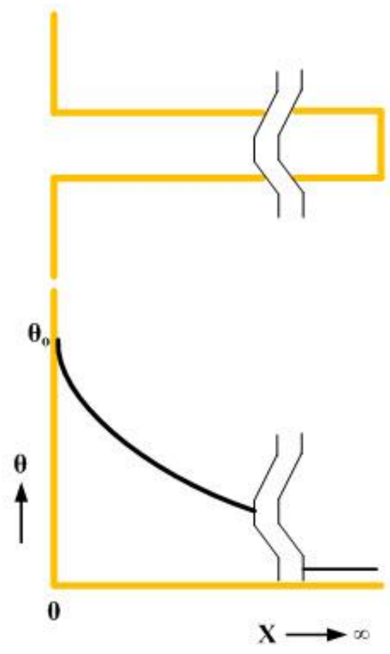



Figure 2

**i) At the base of fin, its temperature is equal to the wall temperature. Therefore,**

At $x = 0$; $T = T_0$

$T - T_a = T_0 - T_a$

or $\theta = \theta_0$ (16)

$\theta_o$ represents temperature difference between the fin base and the fluid surrounding the fin.

**ii) For a sufficient or infinitely long fin, temperature at the tip of fin is equal to that of surroundings.**

At X=L, θ=0 (17)

Applying the first boundary condition to equation (2), we get

$$\theta_o = C_1 \cosh m \times 0 + C_2 \sinh m \times 0$$ , we get

$$C_1 = \theta_0 \qquad (18)$$

Applying second boundary condition to equation (2), we get

$$0 = \theta_o \cosh mL + C_2 \sinh mL$$

$$C_2 = -\theta_o \coth mL \qquad (19)$$

Substituting the values of $C_1$ and $C_2$ in equation (2)

$$\theta = -\theta_o (\cosh mx - \coth mL \sinh mx) \qquad (20)$$

Rate of heat transfer from fin at base or root is expressed as;

$$Q = -kA_c \frac{dT}{dx}|_{x=0} = -kA_c \frac{d\theta}{dx}|_{x=0} \qquad (21)$$

Differentiating equation (20) with respect to x, we get

$$\left(\frac{d\theta}{dx}\right) = \theta_0 (m \sinh mx - \coth mL \times m \cosh mx)$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = \theta_0 (m \sinh m \times 0 - \coth mL \times m \cosh m \times 0)$$

$$\left(\frac{d\theta}{dx}\right)_{x=0} = -\theta_0 m \coth mL \qquad (22)$$

Substituting the value of $\left(\frac{d\theta}{dx}\right)_{x=0}$ in equation (21), we get

$$Q = kA_c \theta_o m \coth mL$$

As $L \to \infty, \coth mL \to 1$

$$\therefore Q = kA_c m\theta_0 \qquad (23)$$

Equation (23) represents rate of heat transfer from a fin of infinite length.

**3. Fin is insulated at the tip**

For a fin of finite length having its end insulated, no heat transfer takes place from the tip of the fin as shown in Figure 3. The relevant boundary conditions are:

**i)** **At** $x = 0; \theta = \theta_0$

**ii)** **At** $x = L; \frac{dT}{dx} = 0 \Rightarrow \frac{d\theta}{dx} = 0$

Applying the first boundary conditions to equation (2), we get

$c_1 = \theta_0$ (24)

Applying second boundary condition to equation (2), we get

$$\left(\frac{d\theta}{dx}\right)_{x=L} = m(\theta_0 \sinh mL + C_2 \cosh mL) = 0$$

$c_2$ = $-\theta_0$ tanh $m$L (25)

Substituting the values of $C_1$ and $C_2$ in equation (2), we get

$$\therefore \theta = \theta_0 \cosh mx - \theta_0 \tanh mL \sinh mx \qquad (26)$$

Equation (26) represents temperature distribution in a fin having its end insulated.

Rate of heat transfer from fin at base or root is expressed as;

$$Q = -kA_c \frac{dT}{dx}\Big|_{x=0} = -kA_c \frac{d\theta}{dx}\Big|_{x=0}$$

$$Q = kA_c \theta_0 m \tanh mL \qquad (27)$$

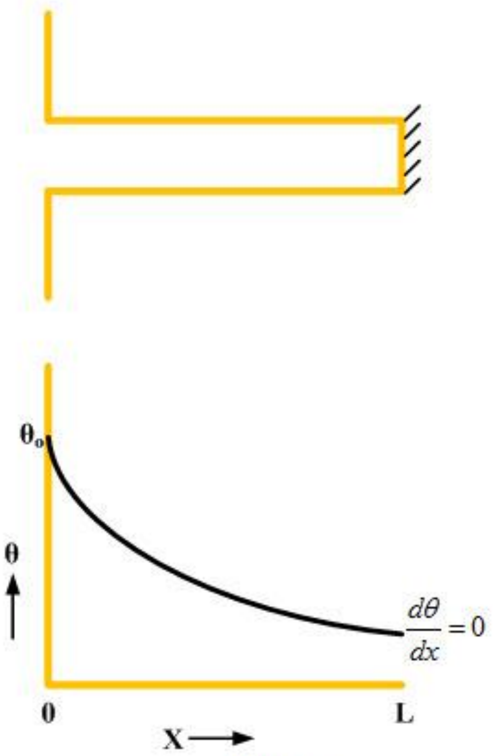



Figure 3

**Lesson 10. Special cases: Fin insulated at the end, fin sufficiently long Variation of Heat Loss from Fins with Length**

## Length of fin

Determination of fin length is an important step in design of a fin once it's cross sectional area has been determined. Longer the fin, larger is the surface area and higher is the heat transfer. However, it is not necessary that the fin should be infinitely long for maximum heat transfer. It has been observed that the temperature drops along the fin exponentially and reaches environmental temperature at some point along the fin length. Beyond this point, length of fin does not contribute to the heat transfer, therefore designing extra long fin results in material waste, excessive weight and increased size at increased cost without any benefit in return.

In order to determine the proper length of the fin, heat transfer from a fin of finite length is compared with that of infinite length.

$$\frac{Q_{finite}}{Q_{infinite}} = \frac{kA_c m\theta_0 \tanh mL}{kA_c m\theta_o} = \tanh mL \qquad (1)$$

Values of hyperbolic function of tanh $mL$ are calculated for some values of $mL$ and it has been found that $Q$ increases linearly initially with but then reaches a plateau for an infinitely long fin for about $mL$ = 5,i.e., $L = \frac{5}{m}$. A fin of length $L = \frac{5}{m}$ is considered as infinitely long. If $\frac{5}{m}$ is reduced to $\frac{2.5}{m}$, there is only 1% reduction in heat transfer. If $L = \frac{1}{m}$, then the fin will transfer about 70% of the heat that can be transferred by an infinitely long fin. Therefore, the length of the fin should be varied from $L = \frac{1}{m}$ to $\frac{2.5}{m}$.

## Effectiveness of fin

The purpose of use of fins is to enhance heat transfer from a surface. Performance of a fin is characterized by fin effectiveness, $\varepsilon_f$ and is defined as ratio of heat transfer from a finned surface to that of without a fin.

$$\varepsilon_f = \frac{\text{Heat transfer from a finned surface}}{\text{Heat transfer from an unfinned surface}} \qquad (2)$$

Consider the case of a rectangular fin that is losing heat at the tip only. Its effectiveness is given expressed as

$$\in = \frac{kA_c m\theta_0 (\frac{mk \sinh mL + h \cosh mL}{h \sinh mL + mk \cosh mL})}{hA_c\theta_0}$$

$$\in = \frac{mk}{h}(\frac{mk \sinh mL + h \cosh mL}{h \sinh mL + mk \cosh mL}) \quad (3)$$

Divide numerator and denominator by

$$\in = \frac{mk}{h}(\frac{mk \tanh mL + h}{h \tanh mL + mk})$$

$$\in = \frac{mk}{h}(\frac{\frac{mk}{h}\tanh mL + 1}{\tanh mL + \frac{mk}{h}}) \quad (4)$$

We know, $m = \sqrt{\frac{Ph}{kA_c}} = \sqrt{\frac{2(w+2\delta)h}{kw2\delta}} = \sqrt{\frac{2wh}{kw2\delta}}$ $\{\because w \rangle\rangle 2\delta\}$

$$\therefore m = \sqrt{\frac{h}{k\delta}} \quad (5)$$

$$\text{BiotNumber, } B_i = \frac{h\delta}{k} = \frac{\delta/k}{1/h} = \frac{\text{Internal resistance of fin material}}{\text{External resistance of fluid on fin surface}} \quad (6)$$

$$\text{Now, } \frac{mk}{h} = \sqrt{\frac{h}{k\delta}}\frac{k}{h} = \sqrt{\frac{hk^2}{k\delta h^2}} = \sqrt{\frac{k}{h\delta}} = \frac{1}{\sqrt{B_i}} \quad (7)$$

$$\text{Also, } mL = \sqrt{\frac{h}{k\delta}}L \times \frac{\delta}{\delta} = \sqrt{\frac{h\delta^2}{k\delta}}\frac{L}{\delta} = \sqrt{\frac{h\delta}{k}}\overline{L} = \sqrt{B_i}\overline{L} \quad (8)$$

{where $\frac{L}{\delta} = \overline{L}$}

Using equations (7) and (8) in equation (4),we get

$$\in = \frac{1}{\sqrt{B_i}}(\frac{\frac{1}{\sqrt{B_i}}\tanh \sqrt{B_i}\overline{L} + 1}{\tanh \sqrt{B_i}\overline{L} + \frac{1}{\sqrt{B_i}}}) \quad (9)$$

Equation (9) gives the expression of effectiveness of a fin in terms of Biot Number, . If , there is no use of putting the fin as heat transfer rate will remain the same. If $B_i \rangle 1, \in \langle 1$, heat transfer rate decreases with the addition of fin as it acts as an insulator. If $B_i \langle 1, \in \rangle 1$, then this is desirable because heat transfer rate increases with the use of a fin.

**Efficiency of fin**

It is generally assumed that bond between a fin and surface, to which fin has been attached, is perfect and it does not offer any thermal resistance. Therefore, temperature at the base of the fin is maximum and equal to the surface temperature. Heat transfer from the base of fin to surrounding fluid will be maximum as temperature difference is maximum at the base of fin and is expressed as

$$Q_{max} = hA_{fin}\,(T_o - T_a) = h\,PL\,(T_o - T_a) \qquad (10)$$

However, due to conductive resistance of fin material, temperature decreases along length of fin, hence, heat transfer rate decreases due to decrease in temperature difference. To account for this decrease in heat transfer, fin efficiency is defined as

$$\eta_f = \frac{\text{Actual heat transfer from fin surface}}{\text{Maximum heat transfer if entire fin surface is maintained at base temperature}}$$

If fin tip is insulated, efficiency is expressed as

$$\eta_f = \frac{kA_c m\theta_o \tanh mL}{hPL\theta_o}$$

$$\eta_f = \frac{kA_c m \tanh mL}{hPL} \qquad (11)$$

For a straight rectangular fin, $A_c = w \times 2\delta$ and $P = 2(w+2\delta)$ as $2\delta << w$, $P=2w$ . Equation (11) becomes

$$\eta_f = \frac{kw \times 2\delta m \tanh mL}{h \times 2wL}$$

$$\eta_f = \frac{mk\delta \tanh mL}{hL} \qquad (12)$$

As $m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h \times 2w}{kw \times 2\delta}} = \sqrt{\frac{h}{k\delta}}$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{h \times 2w}{kw \times 2\delta}} = \sqrt{\frac{h}{k\delta}} \qquad (13)$$

Substituting the value of m from equation (13) in equation (12), we get

$$\eta_f = \sqrt{\frac{hk^2\delta^2}{k\delta h^2 L^2}} \tanh mL$$

$$\eta_f = \frac{1}{L}\sqrt{\frac{k\delta}{h}} \tanh mL$$

Efficiency of most of the fins generally used is above 90 percent.

Problems on fins

- 1 Determine the heat loss per hour by a MS rod of 2 cm diameter and 10 cm length. Temperature at one end of the rod is maintained at 200°C and the temperature of surrounding air is 30°C .The conductivity of the rod is 50 W/m-K and the convective heat transfer coefficient is 20 W/m²- K. Neglect the heat loss at the free end of the rod.
  **Sol** k=50 W/m-K ; h=20 W/m²- K ; L=0.1m ; P=$\pi d$= 0.0628 m ; $A_c = \frac{\pi d^2}{4} = 3.14\times10^{-4}$ m²; $\theta_o = (200-30)K = 170K$
  $m = \sqrt{\frac{Ph}{kA_c}}$ =8.944 ; $Q = km\theta_0 A_c \tanh mL$ =17.03 W=61.323 kJ /hr

- 2 A steel tube is 5 cm in length, 20 cm inner diameter and 25 cm outer diameter .The outer diameter is maintained at 120°C at one end and is exposed to atmosphere at 20°C at the other end. Calculate the heat loss from the tube, neglecting loss of heat from free end and considering that there is no convection from the inside of the tube. Given, conductivity equal to 40 W/m-K and convective heat transfer coefficient equal to 10 W/m²-K.

  **Sol** $d_0$=0.25 m; $d_i$=0.20 m; L=0.05 m; $\theta_0$=100°C; P=$\pi d_o$=0.785 m; $A_c = \frac{\pi}{4}(d_0^2 - d_i^2)$ = 0.0177m²; $m = \sqrt{\frac{Ph}{kA_c}}$ =3.33 ; $Q = km\theta_0 A_c \tanh mL$ =38.9 W=1400.4 kJ /hr

- 3 An Aluminium fin has a cross section of (0.5mm×0.5mm) and is 1 cm long. These fins are provided on the surface of an electronic device. These fins carry 46mW of energy and the temperature of the device should not exceed 80°C. Calculate the number of fins required to do the job if there is no loss of heat from the fin end. Given, k = 190 W/m-K , h = 12.5 W/m²- K and ambient temperature is 40°C. **Sol**
  Length of fin =1cm ; Width of fin =0.5mm ; Thickness of fin =0.5cm ; ⇒P=2×10⁻³m; $A_c$=0.5×0.5×10⁻⁶ m²
  $m = \sqrt{\frac{Ph}{kA_c}}$ =22.94≈23
  $Q = km\theta_0 A_c \tanh mL$ =9.88mW ; ∴Number of fins=46/9.88=4.656≈5

- **4** Find the heat loss from a rod of 4 cm diameter and infinitely long when its base is maintained at 100°C, ambient temperature is 20°C, conductivity is 50 W/m-K and convective heat transfer coefficient is 40 W/m²- K.

  **Sol** $m=\sqrt{\frac{Ph}{kA_c}}$ =8.944 ; $Q=kA_c m\theta_0$ = 44.9W

- **5** Find the heat transferred from an iron fin of thickness 5 mm, height 50 mm, width 100 cm. Also, determine the temperature difference at the tip of the fin, assuming ambient temperature equal to 28°C, conductive heat transfer coefficient as 50 W/m-K and convective heat transfer coefficient as 10 W/m²- K. Temperature difference at base is 80°C.

  **Sol** Length of fin($L$) = 50mm ; Width of fin = 100cm ; Thickness of fin = 5mm ; $T_a$=28°C ; $\theta_0$=80°C ; k=50W/m-K;h=10W/m²-K;P=2(0.5+100)10⁻²m=2.01m; $A_c=5\times10^{-4}$m²

  $m=\sqrt{\frac{Ph}{kA_c}}$ =8.96

  $$Q=kA_c\theta_0 m\left(\frac{mk\sinh mL+h\cosh mL}{h\sinh mL+mk\cosh mL}\right)$$

  $Q$=78.55 W

  $$\text{Also, } \theta=\theta_0\cosh mx-\theta_0\frac{(mk\sinh mL+h\cosh mL)}{(h\sinh mL+mk\cosh mL)}\sinh mx$$

  For $x=L$;=71.9°C ≈ 72°C

**6. Estimate the energy input required to solder together two very long pieces of bare copper wire 0.1625 cm in diameter with a solder that melts at 195°C. the wires are positioned vertically in air at 24°C and the heat transfer coefficient on the wire surface is 17 W/m²-deg. For the wire alloy, take the thermal conductivity 340 W/m-deg.**

**Solution** The physical situation approximates as two infinite fins with a base temperature

of 195°C in an environment at 24°C with the given value of surface coefficient.

Cross-sectional area, $A_c = \frac{\pi}{4}d^2 = \frac{\pi}{4}(0.001625)^2 = 2.073 \times 10^{-6} m^2$

Perimeter, $P = \pi \times 0.001625 = 0.0051\ m$

$$m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{17 \times 0.0051}{335 \times 2.073 \times 10^{-6}}} = 11.17 m^{-1}$$

Heat dissipation from an infinity long fin is

$$Q_{fin} = kA_c m(t_0 - t_a)$$

$$= 340 \times (2.073 \times 10^{-6}) \times 11.17(195 - 24)$$

$$= 1.3228 W$$

Therefore, the energy input required for two wires is **2.645W**

**Example 5.4 A rod of 10mm square section and 160mm length with thermal conductivity of 50 W/m-deg protrudes from a furnace wall at 200°C, and is exposed to air at 30°C with convection coefficient comment on the result. Adopt a long fin model for the arrangement.**

**Solution** Heat dissipation for an infinity long fin is

$Q = kA_c m(t_0 - t_a)$

Where $m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{20\times(4\times0.01)}{50\times(0.01\times0.01)}} = 12.649\ m^{-1}$

Q $=50 \times (0.01 \times 0.01) \times 12.649 \times (200 - 30) = 10.75W$

For the long fin model, the temperature distribution is

$$\frac{\theta}{\theta_0} = \frac{T - T_a}{T_0 - T_a} = e^{-max}$$

At $x = 80\ mm = 0.08\ m$

$$mx = 12.649 \times 0.08 = 1.01192$$

$$\frac{T-30}{200-30} = e^{-1.01192} = 0.3635$$

$$T = 0.3635\times(200 - 30) + 30 = 91.8°C$$

At $x = 158\ m = 0.158\ m\ ; mx = 12.649 \times 0.158 = 1.9985$

$$\frac{T-30}{200-30} = e^{-1.9985} = 0.1355$$

$$T = 0.1355\times(200 - 30) + 30 = 53.04°C$$

Heat conducted upto any length is worked out by taking the difference of total heat and heat conducted at that section.

Heat convected upto 0.08 m length

$= 1075 - k\, A_c m(t_{0.08} - t_a)$

$$=10.75\text{-}50\times (0.01 \times 0.01) \times 12.649(91.8 - 30) = 6.84W$$

Which is $\frac{6.84}{10.75} \times 100 = 63.63\%\ of\ total\ heat\ dissipation.$

Heat convected upto 0.158 m length

$$= 1075 - k\, A_c m(t_{0.08} - t_a)$$

$$=10.75\text{-}50\times (0.01 \times 0.01) \times 12.649(53.04 - 30) = 9.293W$$

Which is $\frac{9.293}{10.75} \times 100 = 86.4\%\ of\ total\ heat\ dissipation.$

**Comments:** Most of heat is dissipated in a short length of the fin. Accordingly it is uneconomical to extend the fin beyond a certain value.

**Example 5.6 Which of the following arrangement of pin fins will give higher heat transfer rate from a hot surface?**

**6 fins of 10 cm length**

**12 fins of 5 cm length**

**The base temperature of the fin is maintained at 200°C and the fin is exposed to a convection environment at 15°C with convection coefficient 25 W/m²-deg. Each fin has cross-sectional area 2.5 cm², perimeter 5 cm and is made of a material having thermal conductivity 250 W/m-deg.**

**Neglect the heat loss from the tip of fin.**

**Solution** The heat loss from n-fins is given by

$Q = n\,k\,A_c\,m(t_0 - t_a)\tanh ml$

$$\text{Where } m = \sqrt{\frac{hP}{kA_c}} = \sqrt{\frac{25\times0.05}{250\times2.5\times10^{-4}}} = 4.472\ m^{-1}$$

**Case I** $n = 6$ and $l = 10\text{cm} = 0.1$ m

$ml = 4.4772\times0.1=0.4472$

$Q_1 = 6[250\times2.5\times10^{-4}\times4.472\times(200-15)\tanh(0.4472)]$

$= 130.18$ W

**Case I** $n = 12$ and $l = 5$ cm $= 0.05$ m

$ml = 4.4772\times0.05=0.2236$

$Q_2 = 12[250\times2.5\times10^{-4}\times4.472\times(200-15)\tanh(0.2236)]$

$= 138.63$ W

The arrangement II is to be preferred as it gives higher rate of heat transfer.

**Example 5.7 An array of 10 fins of anodized aluminium (k=180 W/m-deg) is used to cool a transistor operating at a location where the ambient conditions correspond to temperature 35°C and convective coefficient 12 W/m²-deg. Each fin measures 3 mm wide 0.4 mm thick 5 cm length and has its base at 60°C. Determine the power dissipated by the fin array.**

**Solution** Refer Fig. 5.7 for the fin array. The length of the fin is represented by projection perpendicular to the plane of the pipe.

For a fin of rectangular cross-section,

$$P = 2\,(b + \delta)$$

$$= 2\,(3+0.4) = 608 \text{ mm}$$

$$A_c = b \times \delta = 3 \times 0.4 = 1.2 \text{ mm}^2$$

$$m = \sqrt{\frac{h\,P}{k\,A_c}}$$

$$\sqrt{\frac{12 \times (608 \times 10^{-3})}{180 \times (1.2 \times 10^{-6})}} = 19.44 \text{ m}^{-1}$$

The arrangement corresponds to a fin with tip insulated and for that

$$Q = n\,k\,A_c\,m(t_0 - t_a)\tanh ml$$

$$= 180 \times (1.2 \times 10^{-6}) \times 19.44 \times (60 - 35)\tanh(19.44 \times 0.05)$$

∴ Heat loss from the array of 10 fins,

$$= 0.0786 \times 10 = \mathbf{0.786W}$$

**Example 5.8 A heating unit is made in the form of a vertical tube of 50 mm outside diameter and 1.2 m height. The tube is fitted with 20 steel fins of rectangular section with height 40 mm and thickness 2.5 mm. The temperature at the base of fin is 75° C, the surrounding air temperature is 20°C and the heat transfer coefficient between the fin as well as the tube surface and the surrounding air is 9.5 W/m²K. if thermal conductivity of the fin material is 55W/mK, make calculations for the amount of heat transferred from the tube with and without fin.**

**Solution** Heat flow rate from the tube surface without fin

$$Q_1 = h\ A\ \Delta t = h \times \pi doH \times (t_0 - t_\infty)$$

$$9.5 \times (\pi \times 0.05 \times 1.2) \times (75 - 20) = 98.44\ W$$

(b) Heat flow rate convected from the base

$$Q_b = h\,A_b\,(t_0 - t_\infty)$$

Where $A_b = (\pi \times 0.05 \times 1.2) - 20(1.2 \times .0025) = 0.1284 m^2$

$$\therefore Q_b = 9.5 \times 0.1284 \times (75 - 20) = 67.09\ W$$

Heat flow rate convected from the fins,

$$Q_f = n\ k\ A_c m(t_0 - t_a) \tanh ml$$

Where $A_c$ = cross-sectional area of fin=1.2 × 0.0025 =0.003 $m^2$

P = perimeter of fin = 2 (1.2 + 0.0025) = 2.405 m

$$m = \sqrt{\frac{hp}{kA_c}} = \sqrt{\frac{9.5}{55} \times \frac{2.405}{0.003}} = 11.77$$

Then, $Q_f = 20 \times 55 \times 0.003 \times 11.77 \times (75\text{-}20) \tan h\,(11.77 \times 0.04)$

$= 937.75$ W

∴ Heat flow rate from the take surface when fins are fitted,

$Q_2 = Q_b + Q_f$

$= 67.09 + 937075 = \mathbf{1004.84\ W}$

**Example 5.20 A horizontal steel shaft, 30 mm diameter and 600 mm long, has its first bearing located 100 mm from the end connected to the impeller of a centrifugal pump. If the impeller is immersed in a hot liquid metal at 500° C , work out the temperature at the bearing under the conditions: (a) the shaft is very long (b) the heat flow through the end of the shaft is negligible and (c) the heat is transferred to the surroundings from the end.**

**The temperature and convection coefficient associated with the fluid adjoining the shaft are 35°C and 68 kj/m²-hr-deg. For steel shaft, thermal conductivity k= 72 kj/m-hr-deg.**

**Solution** For the circular shaft

$$\frac{P}{A_c} = \frac{\pi d}{\frac{\pi}{4} d^2} = \frac{4}{d}$$

$$m = \sqrt{\frac{hp}{kA_c}} = \sqrt{\frac{4h}{kd}} = \sqrt{\frac{4 \times 68}{72 \times 0.03}} = 11.22 \text{ m}^{-1}$$

(a) For an infinity long fin

$$\frac{\theta}{\theta_0} = \frac{t_x - t_a}{t_0 - t_a} = e^{-mx}$$

Therefore, temperature $t_x$ at the bearing $(x = 100\ mm = 0.1m)$ is

$$t_x = t_a + (t_0 - t_a)e^{-mx}$$

$$= 35 + (500 - 35)\, e^{-11.22 \times 0.10}$$

$$= 35 + 465 \times 0.3256 = 186.42°C$$

(b) For a fin with no heat loss from the tip end

$$\frac{\theta}{\theta_0} = \frac{t_x - t_a}{t_0 - t_a} = \frac{\cosh m(1-x)}{\cosh ml}$$

Therefore, temperature $t_x$ at the bearing $(x = 100\ mm = 0.1m)$ is

$$t_x = t_a + (t_0 - t_a)\frac{\cosh m(1-x)}{\cosh ml}$$

$$= 35 + (500 - 35)\frac{\cosh 11.22(0.6 - 0.1)}{\cosh(11.22 \times 0.6)}$$

$$= 35 + 465 \times \frac{135.57}{419.41} = \mathbf{186°C}$$

(c) For a fin dissipating heat to the surroundings from its tip end

$$\frac{\theta}{\theta_0} = \frac{t_x - t_a}{t_0 - t_a} = \frac{\cosh m(1-x) + \frac{h}{mk}\sinh m(1-x)}{\cosh ml + \frac{h}{mk}\sinh ml}$$

Therefore, temperature $t_x$ at the bearing ($x = 100\ mm = 0.1m$) is

$$t_x = t_a + (t_0 - t_a)\frac{\cosh m(1-x) + \frac{h}{mk}\sinh m(1-x)}{\cosh ml + \frac{h}{mk}\sinh ml}$$

$$= 35 + (500-35)\frac{\cosh 11.22(0.6-0.1) + \frac{68}{11.22\times 72}\sinh 11.22(0.6-0.1)}{\cosh(11.22\times 0.6) + \frac{68}{11.22\times 72}\sinh(11.22\times 0.6)}$$

$$= 35 + 465\left(\frac{136.57 + 0.084\times 136.57}{419 + 0.084\times 419.41}\right)$$

$$= 35 + 465\times 0.32 = \mathbf{183.8°C}$$

**Example 5.23 A cylinder 5 cm diameter and 1 m long is provided with 10 longitudinal straight fins of material having thermal conductivity 120 W/m-deg. The fins are 0.75 mm thick and protrude 12.5 mm from the cylinder surface. The system is paced in an atmosphere at 40°C and the heat transfer coefficient from the cylinder and fins to the ambient air is 20 W/m$^2$-deg. If the surface temperature of the cylinder is 150°C, calculate the rate of heat transfer and the temperature at the end of fins.**

**Consider the fin to be of finite length**

**Solution** For a fin of rectangular cross-section

$A_c = b \times \delta = 1 \times (0.75 \times 10^{-3}) = 0.75 \times 10^{-3}\ m^2$

$$P = 2(b + \delta) \approx 2\,b = 2 \times 1 = 2\ \mathrm{m}$$

$$m = \sqrt{\frac{hp}{kA_c}} = \sqrt{\frac{20 \times 2}{120 \times 0.75 \times 10^{-3}}} = 21.08\ m^{-1}$$

For a fin of finite length, the heat conducted to the end is convected away to the ambient air and is prescribed by the relation

$$Q = k\,A_c m(t_0 - t_a) \times \frac{\tanh \mathrm{ml} + \frac{h}{km}}{1 + \frac{h}{km} \times \tanh \mathrm{ml}}$$

$$ml = 21.08 \times (12.5 \times 10^{-30}) = 0.2635$$

$$\frac{h}{km} = \frac{20}{120 \times 21.08} = 0.007906$$

$$\tanh \mathrm{ml} = \tanh(0.2635) = 0.2576$$

$$\therefore Q = 120 \times 0.75 \times 10^{-3} \times 21.08 \times (150 - 40) \times \frac{0.2576 + 0.007906}{1 + 0.007906 \times 0.2576}$$

$= 55.29$ W per fin

Heat transfer from unfinned (base surface)

$$= h[\pi d - nA_c]l \times (t_0 - t_a)$$

$$= 20[\pi \times 0.05 - 10 \times (0.75 \times 10^{-3})] \times 1 \times (150 - 40)$$

$$= 328.9\ \mathrm{W}$$

$\therefore Q_{total}$ for 10 fins = 328.9+10×55.29= **881.8** W

(b) For a fin dissipating heat to the surroundings from its tip end

$$\frac{\theta}{\theta_0} = \frac{t_x - t_a}{t_0 - t_a} = \frac{\cosh m(1-x) + \frac{h}{mk} \sinh m(1-x)}{\cosh \mathrm{ml} + \frac{h}{mk} \sinh ml}$$

At the tip ($x = l$), the above identity reduces to

$$\frac{t_l - t_a}{t_0 - t_a} = \frac{1}{\cosh \mathrm{ml} + \frac{h}{mk} \times \sinh ml}$$

Or $$\frac{t_l - 40}{150 - 40} = \frac{1}{\cosh 0.2635 + 0.007906 \times \sinh 0.2635} = 0.964$$

$t_l$ = 40+(150 - 40) ×0.964 = **146.04°C**

**Example 5.26 A steel rod (k= 30 W/m-deg) 1 cm in diameter and 5 cm long protrudes from a wall which is maintained at 100°C. The rod is insulated at its tip and is exposed to an environment with h =50 W/m²-deg and $t_a$=30°C. Calculate the fin efficiency, temperature at the tip of fin and the rate of heat dissipation.**

**Solution** The fin efficiency is given by

$$n_{fin} = \frac{\tan ml}{ml}$$

Where $m = \sqrt{\frac{hp}{kA_c}} = \sqrt{\frac{h \times \pi d}{k\frac{\pi}{4}d^2}} = \sqrt{\frac{4h}{kd}} = \sqrt{\frac{4 \times 50}{30 \times 0.01}} = 25.82m^{-1}$

$$n_{fin} = \frac{\tan(25.82 \times 0.05)}{(25.82 \times 0.05)} = 0.6657\ or\ 66.57\%$$

(b) The temperature distribution for a fin with insulated tip (no heat transfer at the exposed end) is given by the relation

$$\frac{\theta}{\theta_0} = \frac{t_x - t_a}{t_0 - t_a} = \frac{\cosh m(1-x)}{\cosh ml}$$

At tip ($x = l$) and the above equation reduces to

$$\frac{\theta_1}{\theta_0} = \frac{t_1 - 30}{100 - 30} = \frac{1}{\cosh ml}$$

Or $t_1 = 30 + \frac{70}{\cosh(25.82 \times 0.05)} = 30 + 35.79 = \mathbf{65.79°C}$

(c) The heat loss from a fin with insulated tip is

$$Q = kA_c m(t_0 - t_a)\tanh\ ml$$

$$= 30 \times \frac{\pi}{4}(0.01)^2 \times 25.82 \times (100 \times 30) \times \tanh(25.82 \times 005)$$

**=3.658 W**

## Lesson 11. Fin Efficiency and effectiveness, Problems on fins

**Convection:**

As discussed earlier in Lesson-1, transfer of heat between a fluid and a solid surface occurs by convection which is a combination of conduction and mass transport. When a cold fluid comes in contact with a hot solid surface, heat transfer takes place by convection mode and it involves

- Flow of heat from solid surface to fluid particles in contact with it by conduction resulting in increase in temperature and internal energy of fluid particles.
- Movement of heated particles to low temperature region and transfer of heat to low temperature fluid particles.

Therefore, in convection, transfer of heat is dependent on fluid motion which could be laminar or turbulent. In case of laminar flow, heat is transferred between fluid layers by molecular motion on submicroscopic level. In case of turbulent flow, heat transfer mechanism is modified and aided by formation of eddies which result in movement of fluid particles across streamlines. These fluid particles mix with other fluid particles and transfer energy. Heat transfer rate by convection increases with increase in inter mixing or in other words turbulence.

**Free Convection:**

If motion of heated fluid particles occurs due to density differences resulting from temperature variation, heat transfer process is termed as free convection. Heating of a liquid in a vessel placed on a stove is an example of free convection heat transfer process as shown in Figure 1.

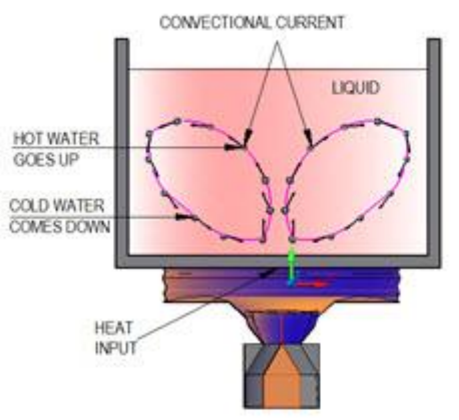



**Figure 1**

**Forced Convection:**

If motion of heated fluid particles is caused by some external source such as a fan, blower or a pump, then heat transfer process is termed as forced convection. The heating of air by blowing it over electric coils in a heater and cooling of air by blowing it over cooling coils in an air conditioner are examples of forced convection as shown in Figure 2.

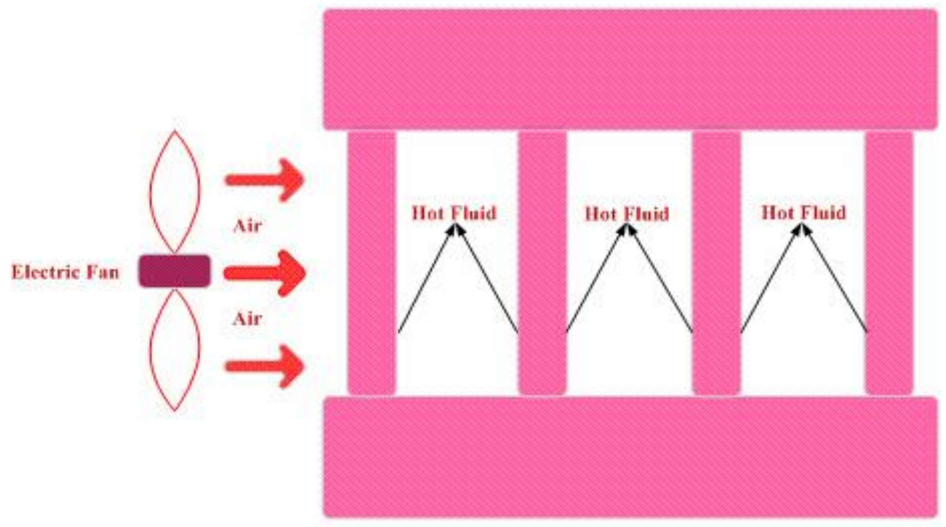



**Figure 2**

**Newton's Law of Cooling:**

During study of cooling of fluids, Newton observed that convection process plays a more prominent role when a fluid and a solid surface exchange heat as compared to pure conduction. He proposed fundamental heat convection equation which is expressed as

$$q = h\,A\,(T_s - T_f) \qquad (1)$$

where

q is average rate of heat transfer by convection, W or J/sec

A is heat transferring area, $m^2$

$T_s$ is temperature of solid surface, $^oC$

$T_f$ is temperature of fluid, $^oC$

h is an average value of proportionality constant called convective heat transfer coefficient, $W/(m^2\text{-}^oC)$

**Convective heat transfer coefficient, h:**

It is not a fluid property but its value depends upon the following parameters

i. Surface geometry

ii. Nature of fluid flow

iii. Bulk fluid velocity

iv. Thermo-physical properties such as thermal density, specific heat, viscosity and thermal conductivity of fluid

Typical values of convective heat transfer coefficient are given in Table 1.

Table 1: Value of Convective Heat Transfer Coefficient ( $W/(m^2\text{-}K)$)

| Type of Convection | Gases | Liquids |
|---|---|---|
| Free Convection | 2-25 | 10-1000 |
| Forced Convection | 25-250 | 50-20,000 |
| Boiling and Condensation | 2500-10000 | |

**Non- Dimensional Numbers:**

i) **Reynolds Number**: Reynolds number is a measure of relative magnitude of inertia force to viscous force occurring in a flow. Higher is the value of Reynolds number, greater will be inertia force and vice-versa. Reynolds number is defined as the ratio of inertia force to viscous force and is expressed as

$$\mathrm{Re} = \frac{Inertia\ force}{Viscous\ force}$$

$$Inertia\ force = Mass \times Acceleration$$

$$= \rho L^3 \times \frac{dV}{dt} = \rho L^3 \times \frac{V}{\frac{L}{V}}$$

$$= \rho L^2 V^2$$

$$Viscous\ force = Shear\ stress \times Area$$

$$= \tau \times L^2 = \mu \frac{\partial V}{\partial y} \times L^2 = \mu \frac{V}{L} \times L^2$$

$$= \mu V L$$

$$\mathrm{Re} = \frac{\rho L^2 V^2}{\mu V L}$$

$$\mathrm{Re} = \frac{\rho L V}{\mu}$$

ii) **Prandtl Number**: It is defined as the ratio of kinematic viscosity to thermal diffusivity and is denoted by 'Pr'.

$$Pr = \frac{Kinematic\ viscosity}{Thermal\ diffusivity}$$

$$Pr = \frac{\upsilon}{\frac{k}{\rho C_p}} = \frac{\rho \upsilon C_p}{k} = \frac{\mu C_p}{k} \qquad \text{as } \rho\upsilon = \mu$$

Kinematic viscosity indicates impulse transport through molecular friction and thermal diffusivity indicates heat transport through conduction. Prandtl number signifies the relative speed with which momentum and energy are propagated through a fluid.

iii) **Nusselt Number:** It is defined as the ratio of heat flow rate by convection under unit temperature gradient to the heat flow rate by conduction under unit temperature gradient through a stationary thickness 'L'.

$$Nu = \frac{h}{\frac{k}{L}} = \frac{hL}{k}$$

iv) **Stanton Number:** Stanton number is defined as ratio of Nusselt number to the product of Reynolds and Prandtl numbers and is denoted by 'St'.

$$St = \frac{\text{Nusselt number}}{\text{Reynolds numeber} \times \text{Prandtl number}} = \frac{Nu}{Re \times Pr}$$

v) **Peclet Number:** Peclet number is defined as ratio of mass heat flow rate to heat flow rate by conduction under unit temperature gradient through a thickness of 'L'.

Mass heat flow rate = Heat capacity of fluid passing through a unit cross-sectional area per unit time

$$= (1 \times V)\ \rho C_p$$

Heat flow rate by conduction under unit temperature gradien $= (k/1) \times (1/L)$

$$Pe = \frac{\rho V C_p}{\frac{k}{L}} = \frac{\rho C_p}{k} LV = \frac{LV}{\alpha}$$

Where $\alpha$ is thermal diffusivity.

$$Pe = \frac{\rho C_p}{k} LV \frac{\mu}{\mu} = \frac{\rho LV}{\mu} \frac{\mu C_p}{k} = Re \times Pr$$

vi) **Gratez Number:** It is defined as the ratio of heat capacity of a fluid flowing through a pipe per unit length to the conductivity of the pipe material.

$$Gr = \frac{\frac{mC_p}{L}}{k} = \frac{mC_p}{kL} = \frac{\rho AVC_p}{kL}$$
$$= \frac{\rho\pi D^2 V}{4kL}$$

vii) **Grashoff Number:** It is defined as the ratio of product of inertia force and buoyancy force to square of viscous force.

$$Gr = \frac{Inertia\ force \times Bouyancy\ force}{(Viscous\ force)^2}$$

$$Gr = \frac{Inertia\ force \times Bouyancy\ force}{(Viscous\ force)^2}$$
$$= \frac{\rho V^2 \times (\rho\beta g\Delta\theta L^3)}{(\mu V)^2}$$
$$= \frac{\rho^2 \beta g\Delta\theta L^3}{\mu^2}$$

Where

$\beta = \frac{1}{T_f}$ and $T_f$ is mean fluid temperature

**Lesson 12. Free and Forced Convection- Newton's law of cooling, heat transfer coefficient in convection, Useful non dimensional numbers**

**Dimensional Analysis for Free and Forced Convection:**

In a number of engineering applications involving flow of fluids over a flat plate, inside and outside of cylinders, heat is exchanged between fluids and solid surfaces. In order to determine heat transfer rate, value of convective heat transfer coefficient must be determined. The following methods are generally used to determine the value of convective heat transfer coefficient.

i) Dimensional Analysis

ii) Solution of Boundary Layer Equations

iii) Analogy between Heat and Momentum Transport

**Dimensional analysis:**

The method of dimensional analysis was first used by Nusselt to derive mathematical equations for convective heat transfer coefficients for free and forced convection. Dimensional analysis is a mathematical technique which is used to obtain equations governing an unknown physical phenomenon in terms of important parameters influencing that phenomenon. The influencing parameters are organized into dimensionless groups, thereby, reducing the number of influencing parameters. Dimensional analysis for free and forced convection involves following steps

i) Determination of all parameters/variables affecting convective heat transfer coefficient.

ii) Writing influencing parameters in terms of fundamental units of mass, length, time and temperature.

iii) Developing mathematical expressions for convective heat transfer coefficient in terms of fundamental units by using principle of dimensional homogeneity.

iv) Grouping of all influencing parameters into non-dimensional numbers.

The dimensions of various parameters which are important from heat transfer point of view are given in terms of fundamental units of mass, length, time and temperature in Table 1.

| Parameter | Symbol | SI Units | Dimensions |
|---|---|---|---|
| Mass | M | Kg | M |
| Length | L | m | L |
| Time | T | Second | T |

| | | | |
|---|---|---|---|
| Temperature | Θ | K | θ |
| Heat | Q | Joule | $ML^2T^{-2}$ |
| Area | A | $m^2$ | $L^2$ |
| Volume | V | $m^3$ | $L^3$ |
| Velocity | U, V | m/sec | $LT^{-1}$ |
| Acceleration | A | m/sec$^2$ | $LT^{-2}$ |
| Acceleration due to Gravity | G | m/sec$^2$ | $LT^{-2}$ |
| Force or Resistance | F, R | N | $MLT^{-2}$ |
| Density | P | Kg/m$^3$ | $ML^{-3}$ |
| Dynamic viscosity | M | Kg/(m-sec) | $ML^{-1}T^{-1}$ |
| Kinematic viscosity | Y | m$^2$/sec | $L^2T^{-1}$ |
| Energy, Work, Heat | E, W | m N | $ML^2T^{-2}$ |
| Convective heat transfer coefficient | H | W/(m$^2$-deg) | $MT^{-3}\ \theta^{-1}$ |
| Coefficient of volumetric expansion | B | Per deg | $\theta^{-1}$ |
| Specific Heat | $C_p$, $C_v$ | kJ/(kg-deg) | $L^2T^2\ \theta^{-1}$ |
| Thermal conductivity | K | W/(m-deg) | $MLT^{-3}\ \theta^{-1}$ |
| Thermal resistance per unit area | $R_t$ | m$^2$-hr- $^o$C/kJ | $M^{-1}T^{-3}\ \theta^{-1}$ |
| Thermal Diffusivity | A | m$^2$/sec | $L^2T^{-1}$ |

**Methods of Dimensional Analysis:**

If number of variables influencing convective heat transfer coefficient are known, then the following two methods can be used to develop a mathematical expression relating the variables with the convective heat transfer coefficient.

i) Rayleigh's Method

ii) Buckingham's π-theorem

However, in application of dimensional analysis for determining convective heat transfer coefficient for free and forced convection, Rayleigh's method will not be used as it has certain limitations that can be overcome by using Buckingham's π-theorem method.

**Buckingham's π-Theorem Method**

In the Rayleigh's method of dimensional analysis, solution becomes more and more cumbersome and laborious if number of influencing variables become more than the fundamental units (M, L, T and θ) involved in the physical phenomenon.. The use of Buckingham's π-theorem method enables to overcome this limitation and states that if there are 'n' variables (independent and dependent) in a physical phenomenon and if these variables contain 'm' number of fundamental dimensions (M, L, T and θ), then the variables are arranged in to (n-m) dimensionless terms called π-terms.

Buckingham's π-Theorem Method can be applied for forced and free convection processes to determine the heat transfer coefficient.

**Dimensional Analysis for Forced Convection**

On the basis of experience, it is concluded that forced convection heat transfer coefficient is a function of variables given below in Table -2

| S. No. | Variable / Parameter | Symbol | Dimensions |
|---|---|---|---|
| 1 | Fluid density | $\rho$ | $ML^{-3}$ |
| 2 | Dynamic viscosity of fluid | $\mu$ | $ML^{-1}T^{-1}$ |
| 3 | Fluid Velocity | V | $LT^{-1}$ |
| 4 | Thermal conductivity of fluid | k | $MLT^{-3}\,\theta^{-1}$ |
| 5 | Specific heat of fluid | $C_p$ | $L^2T^{-2}\,\theta^{-1}$ |
| 6 | Characteristic length of heat transfer area | D | L |

Therefore, convective heat transfer coefficient is expressed as

$h = f(\rho, \mu, V, k, C_p, D)$ (1)

$f(h, \rho, \mu, V, k, C_p, D) = 0$ (2)

Convective heat transfer coefficient, h is dependent variable and remaining are independent variables.

Total number of variables, n = 7

Number of fundamental units, m = 4

According to Buckingham's π-theorem, number of π-terms is given by the difference of total number of variables and number of fundamental units.

Number of π-terms = (n-m) = 7-4 = 3

These non-dimensional π-terms control the forced convection phenomenon and are expressed as

$f(\pi_1, \pi_2, \pi_3) = 0$ (3)

Each π-term is written in terms of repeating variables and one other variable. In order to select repeating variables following method should be followed.

- Number of repeating variables should be equal to number of fundamental units involved in the physical phenomenon.
- Dependent variable should not be selected as repeating variable.
- The repeating variables should be selected in such a way that one of the variables should contain a geometric property such as length, diameter or height. Other repeating variable should contain a flow property such as velocity or acceleration and the third one should contain a fluid property such as viscosity, density, specific heat or specific weight.
- The selected repeating variables should not form a dimensionless group.
- The selected repeating variables together must have same number of fundamental dimensions.
- No two selected repeating variables should have same dimensions.

The following repeating variables are selected

**i) Dynamic viscosity, μ having fundamental dimensions $ML^{-1}T^{-1}$**

**ii) Thermal conductivity, k having fundamental dimensions $MLT^{-3}\theta^{-1}$**

**iii) Fluid velocity, V having fundamental dimensions $LT^{-1}$**

**iv) Characteristic length, D having fundamental dimensions L**

Each π-term is expressed as:

$\pi_1 = \mu^a\, k^b, V^c, D^d, h$ (4)

Writing down each term in above equation in terms of fundamental dimensions

$M^0L^0T^0\,\theta^0 = (ML^{-1}T^{-1})^a\,(MLT^{-3}\,\theta^{-1})^b\,(LT^{-1})^c\,(L)^d\,MT^{-3}\,\theta^{-1}$

Comparing the powers of M, we get

$0 = a+b+1, \quad a+b = -1$ (5)

Comparing powers of L, we get

$0 = -a+b+c+d$ (6)

Comparing powers of T, we get

$0 = -a - 3b - c - 3$ (7)

Comparing powers of θ, we get

$0 = -b - 1,$

$b = -1$ (8)

Substituting value of 'b' from equation (8) in equation (5), we get

$a = 0$ (9)

Substituting values of 'a' and 'b' in equation (7), we get

$c = 0$ (10)

Substituting the values of 'a', 'b' and 'c' in equation (6), we get

$d = 1$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (4), we get

$\pi_1 = \mu^{-0}\, k^{-1}, V^0, D^1, h$

$\pi_1 = h\, D / k$ (11)

The second π –term is expressed as

$\pi_2 = \mu^a\, k^b, V^c, D^d, \rho$ (12)

$M^0L^0T^0\,\theta^0 = (ML^{-1}T^{-1})^a\,(MLT^{-3}\,\theta^{-1})^b\,(LT^{-1})^c\,(L)^d\,ML^{-3}$

Comparing the powers of M, we get

$0 = a+b+1, \quad a+b = -1$ (13)

Comparing powers of L, we get

$0 = -a + b + c + d - 3$ (14)

Comparing powers of T, we get

$0 = -a - 3b - c$ (15)

Comparing powers of θ, we get

$0 = -b\,,$

$b = 0$ (16)

Substituting value of 'b' from equation (16) in equation (13), we get

$a = -1$ (17)

Substituting values of 'a' and 'b' in equation (15), we get

$c = 1$ (18)

Substituting the values of 'a', 'b' and 'c' in equation (14), we get

$d = 1$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (12), we get

$\pi_2 = \mu^{-1} k^0, V^1, D^1, \rho$

$\pi_2 = \rho VD / \mu$ (19)

The third π –term is expressed as

$\pi_3 = \mu^a k^b, V^c, D^d, C_p$ (20)

$M^0L^0T^0 \theta^0 = (ML^{-1}T^{-1})^a (MLT^{-3} \theta^{-1})^b (LT^{-1})^c (L)^d L^2T^{-2} \theta^{-1}$

Comparing the powers of M, we get

$0 = a+b, \quad a+b= 0$ (21)

Comparing powers of L, we get

$0 = -a+ b+ c +d +2$ (22)

Comparing powers of T, we get

$0 = -a- 3b-c-2$ (23)

Comparing powers of θ, we get

$0 = -b -1,$

$b=-1$ (24)

Substituting value of 'b' from equation (24) in equation (21), we get

$a = 1$ (25)

Substituting values of 'a' and 'b' in equation (23), we get

$c = 0$ (26)

Substituting the values of 'a', 'b' and 'c' in equation (22), we get

$d = 0$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (12), we get

$\pi_3 = \mu^1 k^{-1}, V^0, D^0, C_p$

$\pi_3 = \mu C_p/k$ (27)

Substituting the values of $\pi_1$, $\pi_2$, $\pi_3$ in equation (3), we get

$f(h D / k, \rho VD / \mu, \mu C_p/k) = 0$

$h D / k = \varphi(\rho VD / \mu, \mu C_p/k)$

$Nu = \varphi(Re, Pr)$

The above correlation is generally expressed as

$Nu = C (Re)^a (Pr)^b$

The constant C and exponents 'a' and 'b' are determined through experiments.

**Dimensional Analysis for Free Convection:**

In free convection heat transfer process, convective heat transfer coefficient depends upon the same parameters/variable as in case of forced convection except velocity of fluid. It is on account of the fact that in free convection motion of fluid occurs due to difference in density of various layers of fluid caused by temperature difference whereas in case of forced convection motion of fluid is caused by an external source. The fluid velocity in case of free convection depends upon the following parameters;

**i) Temperature difference between solid surface and bulk fluid, $\Delta T$**

**ii) Acceleration due to gravity, g**

**iii) Coefficient of volumetric expansion of fluid, $\beta$**

The change in the volume when temperature changes can be expressed as

$dV = V_1 \beta (T_2 - T_1)$

where

dV - change in volume ($m^3$)

$= V_2 - V_1$

$\beta$ = Coefficient of volumetric expansion of fluid,($m^3/m^3\,^oC$)

$T_2$ - Final temperature ($^oC$)

$T_1$ - Initial temperature ($^oC$)

Therefore, free convection heat transfer coefficient is a function of variables given in Table 3

Table 3

| S. No. | Variable | Symbol | Dimensions |
|---|---|---|---|
| 1 | Fluid density | $\rho$ | $ML^{-3}$ |
| 2 | Dynamic viscosity of fluid | $\mu$ | $ML^{-1}T^{-1}$ |
| 3 | Thermal conductivity of fluid | k | $MLT^{-3}\,\theta^{-1}$ |
| 4 | Specific heat of fluid | $C_p$ | $L^2T^{-2}\,\theta^{-1}$ |
| 5 | Characteristic length of heat transfer area | D | L |
| 6 | Temperature difference between surface and bulk fluid | $\Delta T$ | $\theta$ |
| 7 | Coefficient of volumetric expansion | $\beta$ | $\theta^{-1}$ |
| 8 | Acceleration due to gravity | g | $LT^{-2}$ |

Therefore, convective heat transfer coefficient is expressed as

$$h = f(\rho, \mu, k, C_{p,} D, \Delta T, \beta, g) \qquad (28)$$

However, in free convection, ($\Delta T\ \beta\ g$) will be treated as single parameter as the velocity of fluid particles is a function of these parameters. Therefore, equation (28) can be expressed as

$$f(h, \rho, \mu, k, C_{p,} D, (\Delta T\ \beta\ g)) = 0 \qquad (29)$$

Convective heat transfer coefficient, h is dependent variable and remaining are independent variables.

Total number of variables, n = 7

Number of fundamental units, m = 4

According to Buckingham's π-theorem, number of π-terms is given by the difference of total number of variables and number of fundamental units.

Number of π-terms = (n-m) = 7-4 = 3

These non-dimensional π-terms control the forced convection phenomenon and are expressed as

$f(\pi_1, \pi_2, \pi_3) = 0$ (30)

Each π-term is written in terms of repeating variables and one other variable and the following repeating variables are selected

**i) Dynamic viscosity, μ having fundamental dimensions $ML^{-1}T^{-1}$**

**ii) Thermal conductivity, k having fundamental dimensions $MLT^{-3}\theta^{-1}$**

**iii) Fluid density, ρ having fundamental dimensions $ML^{-3}$**

**iv) Characteristic length, D having fundamental dimensions L**

Each π-term is expressed as:

$\pi_1 = \mu^a\, k^b, \rho^c, D^d, h$ (31)

Writing down each term in above equation in terms of fundamental dimensions

$M^0L^0T^0\,\theta^0 = (ML^{-1}T^{-1})^a\,(MLT^{-3}\,\theta^{-1})^b\,(ML^{-3})^c\,(L)^d\,MT^{-3}\,\theta^{-1}$

Comparing the powers of M, we get

$0 = a+b+c+1, \quad a+b+c = -1$ (32)

Comparing powers of L, we get

$0 = -a+b+c+d$ (33)

Comparing powers of T, we get

$0 = -a-3b-c-3$ (34)

Comparing powers of θ, we get

$0 = -b-1,$

$b = -1$ (35)

Substituting value of 'b' from equation (35) in equation (32), we get

$a = 0$ (36)

Substituting values of 'a' and 'b' in equation (34), we get

$c = 0$ (37)

Substituting the values of 'a', 'b' and 'c' in equation (33), we get

$d = 1$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (31), we get

$\pi_1 = \mu^a\, k^b, \rho^c, D^d, \rho$

$\pi_1 = h\,D / k$ (38)

The second π –term is expressed as

$\pi_2 = \mu^a\, k^b, \rho^c, D^d, C_p$ (39)

$M^0L^0T^0\,\theta^0 = (ML^{-1}T^{-1})^a\,(MLT^{-3}\,\theta^{-1})^b\,(ML^{-3})^c\,(L)^d\,L^2T^{-2}\,\theta^{-1}$

Comparing the powers of M, we get

$0 = a+b+c$ (40)

Comparing powers of L, we get

$0 = -a+ b-3c +d +2$ (41)

Comparing powers of T, we get

$0 = -a- 3b-2$ (42)

Comparing powers of θ, we get

$0 = -b-1, \quad b=-1$ (43)

Substituting value of 'b' from equation (43) in equation (40), we get

$a = 1$ (44)

Substituting values of 'a' and 'b' in equation (42), we get

$c = 0$ (45)

Substituting the values of 'a', 'b' and 'c' in equation (41), we get

$d = 0$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (39), we get

$\pi_2 = \mu^1\, k^{-1}, \rho^0 , D^0, C_p$

$\pi_2 = \mu\, C_p / k$ = Prandtl Number = Pr (46)

The third π –term is expressed as

$\pi_3 = \mu^a\, k^b, \rho^c, D^d, ((\Delta T\,\beta\, g)$ (47)

$M^0L^0T^0\,\theta^0 = (ML^{-1}T^{-1})^a\,(MLT^{-3}\,\theta^{-1})^b\,(ML^{-3})^c\,(L)^d\,(\theta^{-1}LT^{-2}\,\theta^1)$

$M^0L^0T^0\,\theta^0 = (ML^{-1}T^{-1})^a\,(MLT^{-3}\,\theta^{-1})^b\,(ML^{-3})^c\,(L)^d\,(LT^{-2})$

Comparing the powers of M, we get

$0 = a+b+c, \quad a+b+c= 0$ (48)

Comparing powers of L, we get

$0 = -a+b-3c +d +1$ (49)

Comparing powers of T, we get

$0 = -a- 3b-2$ (50)

Comparing powers of θ, we get

$0 = -b, b=0$ (51)

Substituting value of 'b' from equation (51) in equation (48), we get

$a = -2$ (52)

Substituting values of 'a' and 'b' in equation (50), we get

$c = 2$ (53)

Substituting the values of 'a', 'b' and 'c' in equation (49), we get

$d = 3$

Substituting the values of 'a', 'b', 'c' and 'd' in equation (47), we get

$\pi_3 = \mu^{-2} k^0, \rho^2, D^3, (\Delta T\ \beta\ g)$

$\pi_3 = \rho^2 D^3 (\Delta T\ \beta\ g) / \mu^2$

$= D^3 (\Delta T\ \beta\ g) / \upsilon^2$ (54)

Substituting the values of $\pi_1, \pi_2, \pi_3$ in equation (30), we get

$f(h D / k, \mu C_p/k, D^3 (\Delta T\ \beta\ g) / \upsilon^2 ) =0$

$h D / k = \varphi(\mu C_p/k, D^3 (\Delta T\ \beta\ g) / \upsilon^2 )$

$Nu = \varphi(Pr, Gr)$ as $Gr = D^3 (\Delta T\ \beta\ g) / \upsilon^2$ (55)

The above correlation is generally expressed as

$Nu = C (Pr)^a (Gr)^b$ (56)

The constant C and exponents a and b are determined through experiments.

## Lesson 13. Dimensional analysis of free and forced convection

### Empirical Relations for Free and Forced Convection

### 1. Free Convection:

In case of free convection, heat transfer coefficient or Nusselt is expressed as

$$Nu_a = h L / k = f(Gr, Pr)$$

Where

$Nu_a$ is average Nusselt Number

Gr is Grashoff number

Pr is Prandtl Number

**A) For Vertical Plates and Cylinders**

$$Nu_a = \frac{h_a L}{K} = 0.53\,(Gr\,Pr)^{\frac{1}{4}} \qquad \text{for } (Gr\,Pr) < 10^5$$

$$Nu_a = \frac{h_a L}{K} = 0.56\,(Gr\,Pr)^{\frac{1}{4}} \qquad \text{for } 10^5 < (Gr\,Pr) < 10^8$$

$$Nu_a = \frac{h_a L}{K} = 0.13\,(Gr\,Pr)^{\frac{1}{3}} \qquad \text{for } 10^8 < (Gr\,Pr) < 10^{12}$$

where

L is Characteristic length and it is the height of the plate or cylinder

$h_a$ is average heat transfer coefficient.

Gr is Grashoff Number

**B) Horizontal Cylinders**

$$Nu_a = \frac{h_a L}{K} = 1.1\,(Gr\,Pr)^{\frac{1}{6}} \qquad \text{for } \frac{1}{10} < (Gr\,Pr) < 10^4$$

$$Nu_a = \frac{h_a L}{K} = 0.53\,(Gr\,Pr)^{\frac{1}{4}} \qquad \text{for } 10^4 < (Gr\,Pr) < 10^9$$

$$Nu_a = \frac{h_a L}{K} = 0.13\,(Gr\,Pr)^{\frac{1}{3}} \qquad \text{for } 10^9 < (Gr\,Pr) < 10^{12}$$

Where

L is Characteristic length and in this case it is the diameter of the cylinder

$h_a$ is average heat transfer coefficient.

Gr is Grashoff Number

**C) Horizontal Square or Circular Plates**

- For horizontal hot surface facing upward or cold surface facing downward.

$$Nu_a = \frac{h_a L}{K} = 0.71\,(Gr\,Pr)^{\frac{1}{4}} \qquad \text{for } 10^3 < (Gr\,Pr) < 10^9$$

$$Nu_a = \frac{h_a L}{K} = 0.17\,(Gr\,Pr)^{\frac{1}{3}} \qquad \text{for } (Gr\,Pr) > 10^9$$

- For horizontal hot surface facing downward or cold surface facing upward.

$$Nu_a = \frac{h_a L}{K} = 0.35\,(Gr\,Pr)^{\frac{1}{4}} \qquad \text{for } 10^3 < (Gr\,Pr) < 10^9$$

$$Nu_a = \frac{h_a L}{K} = 0.08\,(Gr\,Pr)^{\frac{1}{3}} \qquad \text{for } (Gr\,Pr) > 10^9$$

Where

L is Characteristic length and in case of square plate it is the side of the plate

L is Characteristic length and in case of circular plate it is the diameter

$h_a$ is average heat transfer coefficient.

Gr is Grashoff Number

The properties of the fluid should be calculated at the temperature $\frac{T_s + T_f}{2}$

Where $T_s$ = Plate surface temperature $T_f$ = Fluid temperature.

**2. Forced Convection**

In case of forced convection, heat transfer coefficient or Nusselt is expressed as

$$Nu_x = f\,(x^*, Re_x, Pr)$$

Subscript 'x' has been added to emphasize our interest in conditions at a particular location on the surface.

Where

$Nu_x$ is local Nusselt Number

$Re_x$ is local Reynolds Number

$x^* = \frac{x}{L}$ is dimensionless distance

$Nu_a = f(Re_L, Pr)$

Subscript 'a' indicates an average distance from $x^* = 0$ to the location of interest.

Where,

$Nu_a$ is average Nusselt Number

$Re_L$ is Reynolds number at the location of interest

(A) **Flow of fluid over a flat surface at constant temperature**

- For laminar flow over flat plate which is valid for $Re_L < 5 \times 10^5$.

$$Nu_a = \frac{h_a L}{K} = 0.664 \ Re_L^{\frac{1}{2}} \ Pr^{\frac{1}{3}}$$

$$Nu_x = \frac{h_x x}{K} = 0.332 \ Re_x^{\frac{1}{2}} \ Pr^{\frac{1}{3}}$$

where $h_a$ is average heat transfer coefficient.

$h_x$ is the local heat transfer coefficient.

$$h_a = \frac{1}{L}\int_0^L h_x dx$$

$$Nu_a = 2\ Nu_x \ \text{and} \ h_a = 2\ h_x$$

- If the flow condition on the flat plate is partly laminar and partly turbulent then for

**i) Only Laminar region**

$$Nu_a = \frac{h_a L}{K} = 0.664 \ Re_L^{\frac{1}{2}} \ Pr^{\frac{1}{3}}$$

$$Nu_x = \frac{h_x x}{K} = 0.332 \ Re_x^{\frac{1}{2}} \ Pr^{\frac{1}{3}}$$

$$h_a = \frac{1}{L}\int_0^L h_x dx$$

where, $h_a$ is average heat transfer coefficient.

$h_x$ is the local heat transfer coefficient.

**ii) Only Turbulent region, which is valid for $Re_L > 5 \times 10^5$,**

$$Nu_a = \frac{h_a L}{K} = 0.037 \ \ Re_L^{0.8} \ \ Pr^{\frac{1}{3}} \quad \text{which is valid for } 5 \times 10^5 < Re_L < 10^7$$

$$Nu_x = \frac{h_x x}{K} = 0.0296 \ \ Re_x^{0.8} \ \ Pr^{\frac{1}{3}} \quad \text{which is valid for } 5 \times 10^5 < Re_L < 10^7$$

$$h_a = \frac{1}{L}\int_0^L h_x dx$$

where $h_a$ is average heat transfer coefficient.

$h_x$ is the local heat transfer coefficient.

**iii) Both Laminar and Turbulent region (mixed flow)**

$$Nu_a = \frac{h_a L}{K} = \left[0.037 \ \ Re_L^{0.8} \ - 871\right] Pr^{\frac{1}{3}} \text{which is valid for } 5 \times 10^5 < Re_L < 10^8$$

The properties of the fluid should be calculated at the temperature $\frac{T_S + T_f}{2}$

Where $T_S$ is plate surface temperature

$T_f$ is fluid temperature

**(B) Fluid is flowing inside the tube or through the annulus**

$$Nu_a = \frac{h_a L}{K} = 0.023 \ \ Re_D^{0.8} \ \ Pr^{0.4}$$

Where L is Characteristic length and in this case, it is the diameter of the pipe

where $2300 < Re_D < 12 \times 10^4$

and $0.7 < Pr < 120$, and $\frac{L}{D} < 60$

The properties of the fluid should be taken at the mean temperature of the fluid $T_f$ defined as: $T_f = \frac{T_s + T_m}{2}$ where $T_m = \frac{T_i + T_o}{2}$

Where $T_i$ and $T_o$ are the inlet and outlet temperatures of the fluid and

$T_s$ is surface temperature of the tube.

**Characteristic Length or Equivalent Diameter ($L_c$ or $D_e$):**

Equivalent diameter is usually expressed by the following equation

$$D_e = \frac{4A_c}{P} = \frac{4 \ \frac{\pi}{4} D^2}{\pi D} = D$$

Where $A_c$ = Cross-sectional Area and P = Perimeter.

So for circular tube $D_e$ = D (inner diameter of the tube). The equivalent diameter is also known as characteristic length. The characteristic lengths of a few geometries are given below.

**1) The fluid is flowing through a rectangular duct as shown in Figure 1, then**

$$L_c = \frac{4A_c}{P} = \frac{4ab}{2(a+b)} = \frac{2ab}{a+b}$$

if a = b, then

$$L_c = \frac{2a^2}{2a} = 2a$$

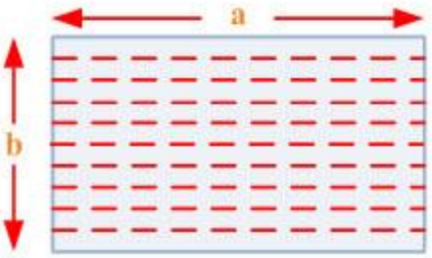



**Figure 1**

**2) If the fluid is flowing through the annulus as shown in Figure 2, then**

$$L_c = \frac{4A_c}{P} = \frac{4}{1} \times \frac{\pi}{4}\frac{\left(D^2 - d^2\right)}{1} \times \frac{1}{\pi\left(D+d\right)} = \left(D - d\right)$$

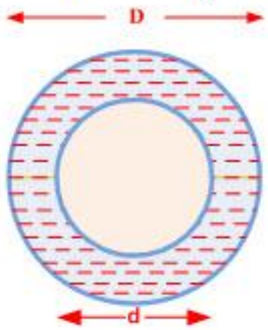



**Figure 2**

**3) If the fluid is flowing through the annulus as shown in Figure 3, then**

$$L_c = \frac{4A_c}{P} = \frac{4(a_1 b_1 - a_2 b_2)}{2[(a_1 + b_1) + (a_2 + b_2)]}$$

If $a_1 = b_1$ and $a_2 = b_2$, then

$$L_c = \frac{4A_c}{P} = \frac{4(a_1^2 - a_2^2)}{(2a_1 + 2a_2)} = (a_1 - a_2)$$

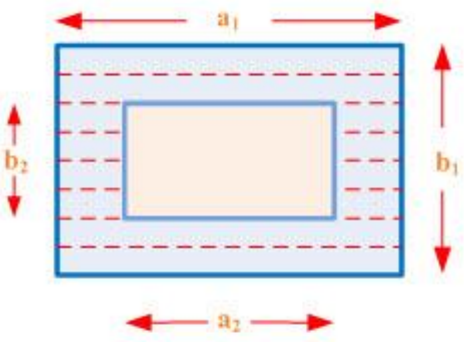



Figure 3

**4) If the fluid is flowing through the annulus as shown in Figure 4, then**

$$L_c = \frac{4A_c}{P} = \frac{4\left(a\ b - \frac{\pi}{4}d^2\right)}{2(a+b) + \pi d}$$

If $a = b$, then

$$L_c = \frac{4a^2}{4a + \pi d}$$

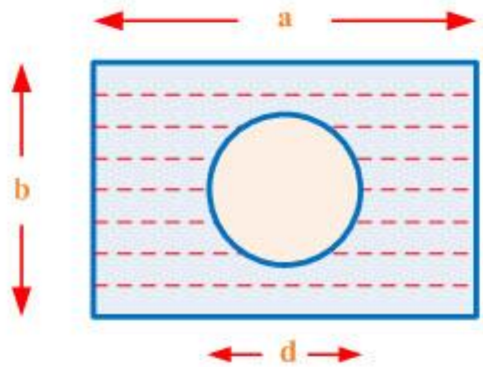


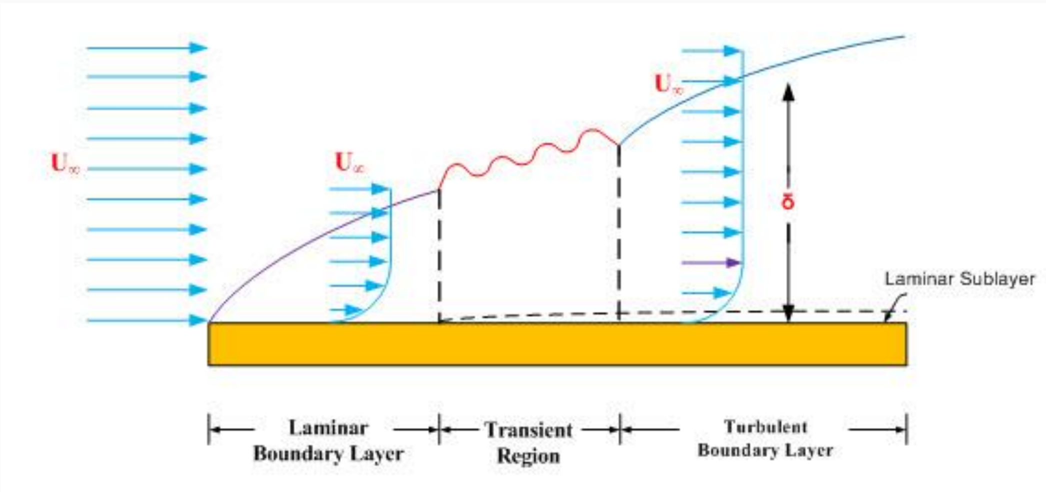

Figure 3

## Lesson 14. Empirical relationships for free and forced convection

**Concept of Boundary Layer:**

Consider a flat plate over which a fluid is flowing in a direction parallel to the plate. The prevailing flow conditions over the plate have following salient features:

i) As a fluid flows over a solid surface, a layer of fluid coming in contact with solid surface sticks to it due to viscosity. This layer of fluid cannot slip away from the solid surface and attains the same velocity as that of the solid surface. If the solid surface is stationary, the velocity of fluid layer sticking to the surface will be zero. This stationary layer of fluid retards the movement of adjacent layer of fluid resulting in development of a small region near the solid surface in which velocity of fluid increases from zero at solid surface to the velocity of main stream. This region is known as hydrodynamic boundary layer. In this region of boundary layer, there is a large variation in velocity of fluid in a direction normal to the solid surface. Figure 1 shows boundary layer for a fluid flowing over a flat plate.




ii) With-in boundary layer: $\dfrac{\partial u}{\partial y} \neq 0$ and $u \neq u_\infty$

Beyond boundary layer: $\dfrac{\partial u}{\partial y} = 0$ *and* $u = u_\infty$

Where u is velocity of flow in boudary layer region and is the free stream velocity.

iii) With the increase in distance from the flat plate in vertical direction, velocity of flow increases and distance at which velocity of flow attains a value of  is called thickness of boundary layer denoted by ‘’. Thickness of boundary layer increases with increase in distance from leading edge in flow direction. The growth of boundary layer i.e. thickness is a function of incoming velocity of flow and kinematic viscosity of fluid. At higher incoming velocities, boundary layer growth is suppressed and a higher value of kinematic viscosity of a fluid results in higher value of boundary layer thickness.

iv) Flow with-in the boundary layer is laminar upto certain distance from the leading edge and beyond this distance; flow becomes unstable and becomes transitional in nature. The flow becomes fully turbulent after passing through a transition zone of finite length. Transition from laminar to turbulent flow in boundary layer region is determined by the value of Reynolds number which is expressed as

$$Re = \frac{u_\infty x}{\upsilon} \tag{1}$$

Where 'x' is distance from leading edge of the plate in flow direction and 'ʋ' is kinematic viscosity.

Laminar flow turns in to turbulent flow in boundary layer region at Reynolds number values between $3*10^5$ to $5*10^5$.

v) For fully turbulent flow in boundary layer region, a laminar sublayer is formed in the immediate vicinity of plate surface where flow is laminar and beyond this sublayer flow is turbulent.

vi) At the plate surface, velocity gradient and shear stress have higher values and along the flow direction these values diminish for laminar boundary layer. However, in turbulent boundary layer, velocity gradient and shear stress have high values near the plate surface.

vii) For flow through pipes, boundary layer growth is similar to that over a flat plate except that thickness of the boundary layer is limited to the radius of the pipe as the flow is taking place in an enclosed space. Boundary layer starts developing from pipe walls, meet at the center of pipe and entire flow acquires characteristics of a boundary layer resulting in fully-developed flow. For fully-developed flow, velocity profile is constant; there is no variation in pressure and velocity of flow. The entrance length required for the flow to turn in to fully-developed turbulent flow is a function of fluid properties, initial level of turbulence, downstream conditions and is generally estimated to be 50-80 times the pipe diameter. The velocity distribution for a fully-developed laminar and turbulent boundary layer in a pipe has been shown in Figure 2.

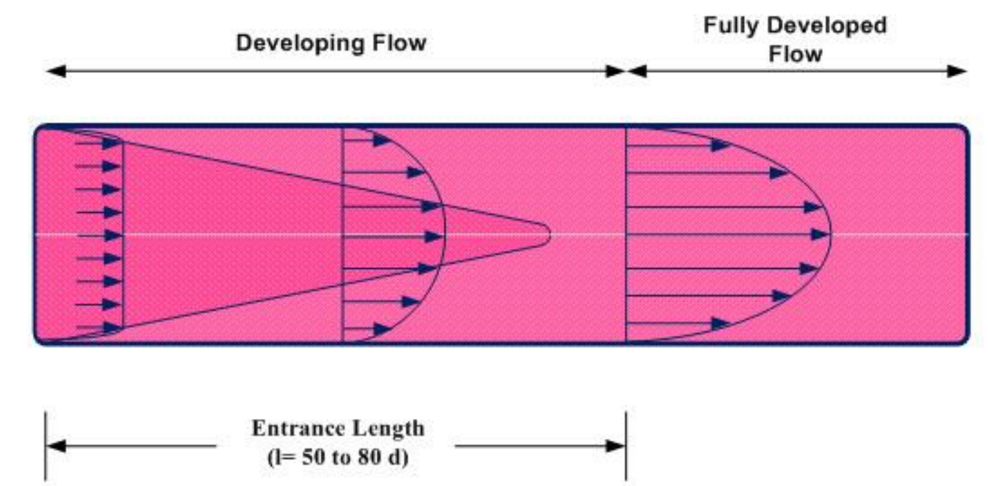



**Figure 2 Development of Boundary Layer in Pipe**

viii) The local skin friction coefficient is defined as ratio of the local wall shear stress to the dynamic pressure of uniform flow.

$$C_{fx} = \frac{\tau_w}{\frac{1}{2}\left(\rho u_\infty^2\right)} \tag{3}$$

ix) Depending upon the nature of velocity profile, boundary layer thickness and skin friction coefficient are expressed as

- For velocity profile $\frac{u}{u_\infty} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ (4)

$$\frac{\delta}{x} = \frac{5.47}{\sqrt{\text{Re}_x}} \tag{5}$$

and $C_{fx} = \frac{1.462}{\sqrt{\text{Re}_x}}$ (6)

- For velocity profile $\frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$ (7)

$$\frac{\delta}{x} = \frac{4.64}{\sqrt{\text{Re}_x}} \tag{8}$$

and $C_{fx} = \frac{1.292}{\sqrt{\text{Re}_x}}$ (9)

- The Blasius technique for an exact solution gives

$$\frac{\delta}{x} = \frac{5}{\sqrt{\text{Re}_x}} \tag{10}$$

$$C_{fx} = \frac{1.328}{\sqrt{\text{Re}_x}} \tag{11}$$

- Reynolds number is expressed as

$$\text{Re} = \frac{\rho u_\infty x}{\mu} \tag{12}$$

- Average skin friction coefficient $\bar{C}_f$ is expressed as:

$$\bar{C}_f = \frac{1.328}{\sqrt{\text{Re}_l}} \tag{13}$$

- Friction force is expressed as

$$Friction\ force = \bar{C}_f \times \frac{1}{2}\rho u_\infty^2 \times area\ of\ plate \tag{14}$$

**Thermal Boundary Layer:**

Similar to hydrodynamic boundary layer, a thermal boundary layer also develops when a fluid flows over a solid surface which is at a temperature different than that of the fluid. Temperature of the fluid layer in contact with the solid surface becomes equal to that of the solid surface. This fluid layer exchanges energy with the adjoining fluid layers resulting in development of region in which temperature variation exists. The region above the solid surface in which temperature variation in the direction normal to the surface exists is called thermal boundary layer. Figure 3 shows thermal boundary layer for a fluid flowing over a solid surface.

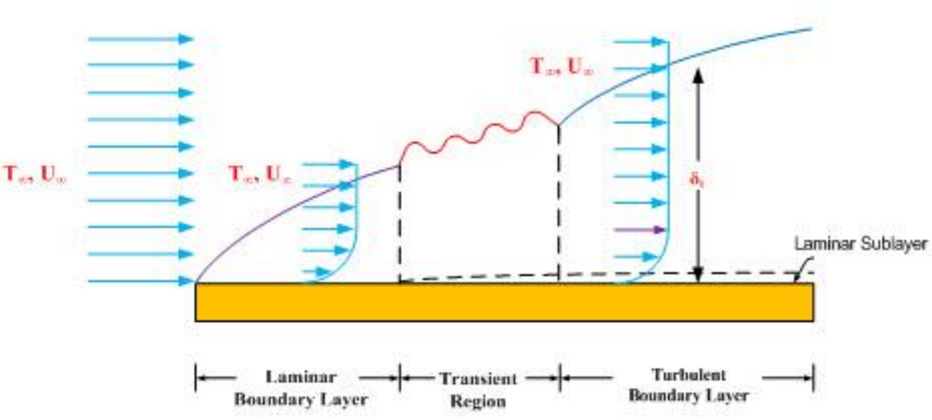



**Figure 3 Thermal Boundary Layer**

The thermal boundary layer thickness ʻʼ is defined as the distance from the solid surface in a direction normal to it at which

$$\frac{T_s - T}{T_s - T_\infty} = 0.99 \qquad (15)$$

Where $T_S$ is the temperature of the surface

$T_\infty$ is the temperature of fluid outside the thermal boundary layer also called free stream temperature.

ʻT is the temperature at a given distance from solid surface with in thermal boundary layer.

At the leading edge of the solid surface, temperature gradient is very high and its value decreases with increase in distance from the solid surface in vertical direction as temperature of fluid approaches free stream temperature. In turbulent boundary layer, formation of eddies decrease the value of temperature gradient.

If the free stream temperature, of the fluid is higher than that of the surface, fluid temperature will be minimum at the surface of the solid and it increases gradually till it becomes equal to the free stream temperature. The vertical distance measured from the surface of the solid to the point where fluid temperature is equal to the free stream temperature denotes the thickness of the thermal boundary layer. Temperature profile in a thermal boundary layer is a function of flow velocity, specific heat, viscosity and conductivity of the fluid. The relationship between thicknesses of hydrodynamic and thermal boundary layers is governed by the Prandtl number value. Prandtl number is expressed as

$$\Pr = \frac{\mu C_p}{k} \tag{16}$$

- For Pr =1, $\delta_t = \delta$

$$\frac{\delta_t}{x} = \frac{5}{\sqrt{\mathrm{Re}_x}} \tag{17}$$

Thicknesses of Thermal and hydrodynamic boundary layer are equal.

- For Pr>1, $\delta_t \prec \delta$

$$\frac{\delta_t}{x} \prec \frac{5}{\sqrt{\mathrm{Re}_x}} \tag{18}$$

Hydrodynamic boundary layer is thicker than the thermal boundary layer.

- For Pr<1, $\delta_t \succ \delta$

$$\frac{\delta_t}{x} \succ \frac{5}{\sqrt{\mathrm{Re}_x}} \tag{19}$$

Thermal boundary layer is thicker than the hydrodynamic boundary layer.

A correlation proposed by Pohlhausen relates Prandtl number with Thermal and hydrodynamic boundary layer thicknesses and is expressed as:

$$\delta_t = \delta \times \Pr^{-0.33} \tag{20}$$

At the surface of the solid as fluid velocity is equal to zero, the heat flux at the solid surface may be expressed as

$$\frac{Q}{A} = h_x (T_s - T_\infty) = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{21}$$

Temperature gradient as suggested by Prandtl is expressed as

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = -\frac{0.332}{x}(T_s - T_\infty)\mathrm{Re}_x^{0.5}\Pr^{0.3} \tag{22}$$

Using equation (22), equation (21) can be expressed as

$$\frac{Q}{A} = h_x (T_s - T_\infty) = 0.332\frac{k}{x}(T_s - T_\infty)\mathrm{Re}_x^{0.5}\Pr^{0.3} \tag{23}$$

$$h_x = 0.332\frac{k}{x}\mathrm{Re}_x^{0.5}\Pr^{0.3} \tag{24}$$

$$\frac{h_x x}{k} = Nu_x = 0.332\,\mathrm{Re}_x^{0.5}\Pr^{0.3} \tag{25}$$

Equations (24) and (25) give the local values of convective heat transfer coefficient and Nusselt number at a given distance from the leading edge of the solid surface. Average values of convective heat transfer coefficient and Nusselt number over the entire length, L of the solid surface is given as

$$h_L = 2 \times 0.332 \frac{k}{L} \mathrm{Re}_L^{0.5} \mathrm{Pr}^{0.3} \tag{26}$$

$$Nu_L = 0.664 \, \mathrm{Re}_L^{0.5} \mathrm{Pr}^{0.3} \tag{27}$$

**Lesson 15. Laminar Forced Convection on a Flat Plate**

Consider a fluid flowing over a flat plate with a velocity U and temperature $T_f$. Let us consider a control volume at a distance 'x' from leading edge of the plate having thickness dx as shown in Figure 1. Following assumptions have been made in order to calculate heat conducted into laminar boundary layer:

i) Thermo-physical properties of the fluid such as thermal conductivity k, specific heat $C_p$ and density ρ remains constant for the range of the temperature

ii) Heating of plate starts from a distance $x_o$ from leading edge of the plate. Within the initial length $x_o$, plate temperature is equal to that of the fluid and there is only hydrodynamic boundary layer and no thermal boundary layer exists. Thermal boundary layer starts developing beyond length $x_o$ and grows beyond that.

iii) Width of the plate is considered to be unity.

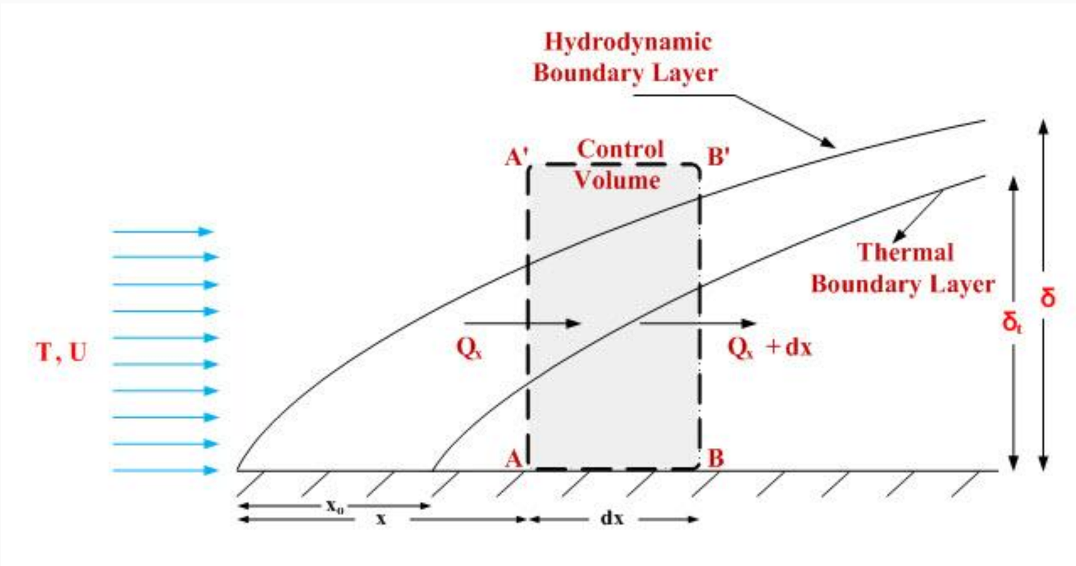



**Figure 1**

Mass of fluid entering into control volume through left face AA'

$$= \,\int\limits_0^H {\rho \,u\,dy} \qquad (1)$$

Mass of fluid leaving control volume through right face BB'

$$=\,\int\limits_0^H {\rho \,u\,dy} \,\, + \,\frac{\partial }{{\partial x}}\left[ {\int\limits_0^H {\rho \,u\,dy} \,} \right]dx \qquad (2)$$

Mass of fluid entering from top face A'B' of control volume

$$= \,\,\frac{\partial }{{\partial x}}\left[ {\int\limits_0^H {\rho \,u\,dy} \,} \right]dx \qquad (3)$$

Heat Influx through face AA'

$${Q_x} = \,\,mass\, \times \,specific\,heat\, \times \,temperature$$

$${Q_x} = \,\int\limits_0^H {\rho \,u\,dy}\times \,C\, \times \,T$$

$$= \,\rho C\,\int\limits_0^H {\,u\,Tdy} \qquad (4)$$

Heat efflux through face BB'

$${Q_{x + dx}}\, = \,\,\int\limits_0^H {\,u\,T} \,\, + \frac{\partial }{{\partial x}}\left[ {\rho C\int\limits_0^H {\,u\,Tdy} } \right]\,dx \qquad (5)$$

The upper face A'B' of the control volume is out of the thermal boundary layer and there temperature is constant at $T_f$. Therefore, energy influx is

$${Q_h} = \,\,\frac{\partial }{{\partial x}}\left[ {\int\limits_0^H {\rho \,u\,dy} \,} \right]dx\,C\,{T_f} \qquad (6)$$

Heat is conducted in to the lower face of the control volume at the rate.

$${Q_c} = \, - kA{\left( {\frac{{\partial T}}{{\partial y}}} \right)_{y = 0}}$$

$${Q_c} = \, - kdx\, \times \,1{\left( {\frac{{\partial T}}{{\partial y}}} \right)_{y = 0}} \qquad (7)$$

An energy balance of the control volume gives:

$$\rho C\,\int\limits_0^H {\,u\,Tdy} \,\, + \frac{\partial }{{\partial x}}\left[ {\rho C\,\int\limits_0^H {\,u{T_f}\,dy} \,} \right]dx\,\, - kdx\,{\left( {\frac{{\partial T}}{{\partial y}}} \right)_{y = 0}} = \,\rho C\,\int\limits_0^H {\,u\,Tdy} \,\, + \frac{\partial }{{\partial x}}\left[ {\rho C\,\int\limits_0^H {\,uT\,dy} \,} \right]dx \qquad (8)$$

Rearranging equation (8), we get,

$$\frac{d}{{dx}}\left[ {\,\int\limits_0^H {\,u\left( {{T_f} - T} \right)\,dy} \,} \right]\,\, = \frac{k}{{\rho C}}\,{\left( {\frac{{dT}}{{dy}}} \right)_{y = 0}}$$

$$\frac{d}{{dx}}\left[ {\,\int\limits_0^H {\,u\left( {{T_f} - T} \right)\,dy} \,} \right]\,\, = \alpha \,{\left( {\frac{{dT}}{{dy}}} \right)_{y = 0}} \qquad (9)$$

Where $\alpha$ represents thermal diffusivity

Equation (9) represents the integral equation for the boundary layer for constant properties and constant free stream temperature $T_f$.

The net viscous work done with in the control volume is given by the equation

$$\mu \,\int\limits_0^H {\frac{{{\partial ^2}u}}{{\partial {y^2}}}} \,dxdy \qquad (10)$$

If the net viscous work done is also considered in the energy balance, then the integral equation would become

$$\frac{d}{{dx}}\left[ {\,\int\limits_0^H {\,u\left( {{T_f} - T} \right)\,dy} \,} \right]\, + \,\frac{\mu }{{\rho C}}\int\limits_0^H {\frac{{{\partial ^2}u}}{{\partial {y^2}}}} \,dy = \frac{k}{{\rho C}}\,{\left( {\frac{{dT}}{{dy}}} \right)_{y = 0}} \qquad (10)$$

The term related to viscous work is generally very small and is usually neglected.

To develop an expression for convective heat transfer coefficient for laminar flow over a plate, cubic velocity and temperature distribution in integral boundary layer equation will be used.

i) The temperature distribution with in the boundary layer is given as

$$\frac{u}{{{u_f}}}\, = \,\frac{3}{2}\left( {\frac{y}{\delta }} \right)\, - \,\frac{1}{2}{\left( {\frac{y}{\delta }} \right)^3}\, \qquad (11)$$

ii) Temperature distribution with in the boundary layer satisfies the conditions;

At $y = 0$, $T = T$

At $y = \delta_t$, $\frac{{dT}}{{dy}}\, = \,0$

At $y = \delta_t$, $T = T_f$

At $y = 0$, $\frac{{{d^2}T}}{{d{y^2}}}\, = \,0$

These boundary conditions have same form as those on $\frac{u}{{{u_f}}}\,$ . Therefore, when these are fitted to a cubic polynomial

$$\frac{\theta }{{{\theta _f}}}\, = \,a\, + \,b\left( {\frac{y}{{{\delta _t}}}} \right)\, + c{\left( {\frac{y}{{{\delta _t}}}} \right)^2}\, + d{\left( {\frac{y}{{{\delta _t}}}} \right)^3} \qquad (12)$$

Temperature distribution acquires the form

$$\frac{\theta }{{{\theta _f}}}\, = \,\frac{{T - {T_s}}}{{{T_f} - {T_s}}} = \frac{3}{2}\left( {\frac{y}{{{\delta _t}}}} \right)\, - \frac{1}{2}{\left( {\frac{y}{{{\delta _t}}}} \right)^3}\, \qquad (13)$$

Multiplying and dividing right hand side of the integral equation by ${u_f}\left( {{T_f} - T} \right)$, we can write

$$\alpha \,{\left( {\frac{{dT}}{{dy}}} \right)_{y = 0}} = {u_f}\left( {{T_f} - T} \right)\frac{d}{{dx}}\left[ {\,\int\limits_0^H {\,\frac{{u\left( {{T_f} - T} \right)}}{{{u_f}\left( {{T_f} - T} \right)}}\,dy} \,} \right]\,$$

$$=\,{u_f}\left( {{T_f} - {T_s}} \right)\frac{d}{{dx}}\left[ {\,\int\limits_0^H {\,\frac{u}{{{u_f}}}\left( {1 - \frac{{\left( {T - {T_s}} \right)}}{{\left( {{T_f} - {T_s}} \right)}}} \right)\,dy} \,} \right]$$

Using equations (12) and (13), we can write

$$\alpha \,{\left( {\frac{{dT}}{{dy}}} \right)_{y = 0}}\, = \,{u_f}\left( {{T_f} - {T_s}} \right)\frac{d}{{dx}}\left[ {\,\int\limits_0^H {\,\left\{ {\frac{3}{2}\left( {\frac{y}{\delta }} \right)\, - \,\frac{1}{2}{{\left( {\frac{y}{\delta }} \right)}^3}\,} \right\}\left\{ {\frac{3}{2}\left( {\frac{y}{{{\delta _t}}}} \right)\, - \frac{1}{2}{{\left( {\frac{y}{{{\delta _t}}}} \right)}^3}\,\,} \right\}\,dy} \,} \right] \qquad (14)$$

For most of the gases thermal boundary layer is thinner than the hydrodynamic boundary layer $\delta_t < \delta$. Therefore the upper limit of integration in equation (14) has been changed to $\delta_t$ as for $y > \delta_t$, the integrand will become zero. Let 'r' represents thickness ratio and it is equal to $\delta_t/\delta$.

Upon integrating equation (14) between limits, we get

$$\alpha \,{\left( {\frac{{dT}}{{dy}}} \right)_{y = 0}}\, = \,{u_f}\left( {{T_f} - {T_s}} \right)\frac{d}{{dx}}\left[ {\delta \,\left( {\frac{3}{{20}}{r^2} - \frac{3}{{280}}{r^4}} \right)} \right] \qquad (15)$$

As $\delta_t < \delta$, $r < 1$, therefore, term involving $r^4$ may be neglected

$$\alpha \,{\left( {\frac{{dT}}{{dy}}} \right)_{y = 0}}\, = \,\frac{3}{{20}}{u_f}\left( {{T_f} - {T_s}} \right)\frac{d}{{dx}}\left[ {\delta {r^2}} \right] \qquad (16)$$

Using temperature distribution equation (13), we can write

$$\frac{{T - {T_s}}}{{{T_f} - {T_s}}} = \frac{3}{2}\left( {\frac{y}{{{\delta _t}}}} \right)\, - \frac{1}{2}{\left( {\frac{y}{{{\delta _t}}}} \right)^3}\,$$

$$T\, = \,{T_s} + \left( {{T_f} - {T_s}} \right)\left[ {\frac{3}{2}\left( {\frac{y}{{{\delta _t}}}} \right)\, - \frac{1}{2}{{\left( {\frac{y}{{{\delta _t}}}} \right)}^3}} \right]$$

$$\frac{{dT}}{{dy}} = \left( {{T_f} - {T_s}} \right)\left[ {\frac{3}{{2{\delta _t}}}\, - \frac{3}{{2\delta _t^3}}{y^2}} \right]$$

$${\left( {\frac{{dT}}{{dy}}} \right)_{y = 0}} = \frac{{3\left( {{T_f} - {T_s}} \right)}}{{2{\delta _t}}} = \frac{{3\left( {{T_f} - {T_s}} \right)}}{{2r\delta }} \qquad (17)$$

Substituting the value of ${\left( {\frac{{dT}}{{dy}}} \right)_{y = 0}}$ from equation (17) in equation (16), we get

$$\frac{3}{2}\alpha \frac{{\left( {{T_f} - {T_s}} \right)}}{{r\delta }} = \frac{3}{{20}}{u_f}\left( {{T_f} - {T_s}} \right)\frac{d}{{dx}}\left[ {\delta {r^2}} \right]$$

$$\alpha= \frac{{{u_f}}}{{10}}\left( {r\delta } \right)\frac{d}{{dx}}\left[ {\delta {r^2}} \right]$$

$$= \frac{u_f}{10}(r\delta)\left(2r\delta\frac{dr}{dx} + r^2\frac{dr}{dx}\right)$$

$$\alpha = \frac{u_f}{10}\left(2r^2\delta^2\frac{dr}{dx} + \delta r^3\frac{dr}{dx}\right) \qquad (18)$$

Using the hydrodynamic boundary layer equations

$$\delta \frac{{d\delta }}{{dx}} = \frac{{140}}{{13}}\frac{v}{{{u_f}}}\,\,and\,{\delta ^2} = \frac{{280}}{{13}}\frac{{vx}}{{{u_f}}}\,$$

Substituting these values in equation (18), we get

$$\alpha = \frac{u_f}{10}\left(2r^2 \frac{280\, vx}{13u_f}\frac{dr}{dx} + r^3 \frac{140\, v}{13u_f}\right)$$

$$r^3 + 4r^2 x\frac{dr}{dx} = \frac{13}{14}\frac{\alpha}{v} \qquad (19)$$

Equation (19) is a linear differential equation of first order in $r^3$ and general solution for it is given as

$${r^3} = C\,{x^{ - \frac{3}{4}}} + \frac{{13}}{{14}}\,\frac{\alpha }{v} \qquad (20)$$

The constant C is determined by using the boundary condition

At $x=x_o$, $${r^3} = {\left( {\frac{{{\delta _t}}}{\delta }} \right)^3} = 0$$

$$0 = C\,x^{-\frac{3}{4}} + \frac{13}{14}\frac{\alpha}{v}, \quad C = \frac{13}{14}\frac{\alpha}{v}x_o^{3/4} \qquad (21)$$

Substituting the value of C from equation (21) into equation (20), we get

$$r^3 = -\frac{13}{14}\frac{\alpha}{v}x_o^{3/4}x^{-\frac{3}{4}} + \frac{13}{14}\frac{\alpha}{v}$$

$$r^3 = \frac{13}{14}\frac{\alpha}{v}\left[1 - \left(\frac{x_o}{x}\right)^{3/4}\right] \qquad (22)$$

Therefore,

$$r = \left(\frac{13}{14}\right)^{1/3}\left(\frac{\alpha}{v}\right)^{1/3}\left[1 - \left(\frac{x_o}{x}\right)^{3/4}\right]^{1/3}$$

$$r = \frac{0.976}{\Pr^{1/3}}\left[1 - \left(\frac{x_o}{x}\right)^{3/4}\right]^{1/3} \qquad (23)$$

If heating of the plate starts from the leading edge of the plate, then $x_o=0$. Equation (23) becomes

$$r = \frac{0.976}{\Pr^{1/3}} \qquad (24)$$

The local heat transfer coefficient can be determined as

$$\frac{Q}{A}={h_x}\left( {{T_s} - {T_f}} \right)=- k{\left( {\frac{{dT}}{{dy}}} \right)_{y = 0}}$$

$$h_x = \frac{-k\left( {dT}/{dy} \right)_{y=0}}{\left(T_s - T_f\right)} \qquad (25)$$

Substituting the value of $\left( {dT}/{dy} \right)_{y=0}$ from equation (17) in to equation (25)

$$h_x = -\frac{3k}{2\delta_t}\frac{\left(T_f - T_s\right)}{\left(T_s - T_f\right)} = \frac{3k}{2\delta_t} = \frac{3k}{2r\delta} \qquad (26)$$

We know that

$$\delta = \frac{4.64\,x}{\sqrt{\mathrm{Re}_x}} \qquad (27)$$

Substituting the values of δ from equation (27) and r from equation (23) in equation (26)

$$h_x = \frac{3k}{2}\frac{\sqrt{\mathrm{Re}_x}}{4.64x} \times \frac{\mathrm{Pr}^{1/3}}{0.976\left[1-\left({x_o}/{x}\right)^{3/4}\right]^{1/3}}$$

$$h_x = 0.332\frac{k}{x}\left(\mathrm{Re}_x\right)^{1/2}\left(\mathrm{Pr}\right)^{1/3} \times \frac{1}{\left[1-\left({x_o}/{x}\right)^{3/4}\right]^{1/3}} \qquad (28)$$

Nusselt Number can be expressed as

$$Nu_x = \frac{x h_x}{k} = 0.332\left(\mathrm{Re}_x\right)^{1/2}\left(\mathrm{Pr}\right)^{1/3} \times \frac{1}{\left[1-\left({x_o}/{x}\right)^{3/4}\right]^{1/3}}$$

If the entire length of the plate is heated, $x_o=0$

$$Nu_x = 0.332(\mathrm{Re}_x)^{1/2}(\mathrm{Pr})^{1/3} \qquad (29)$$

$$h_x = 0.332\frac{k}{x}(\mathrm{Re}_x)^{1/2}(\mathrm{Pr})^{1/3} \qquad (30)$$

**Problem 11.1 Air at atmospheric pressure and 90° C flows with a velocity of 8 m/s across a tube of 25 mm diameter. The tube is maintained at a temperature of 650° C. Calculate the rate of heat transfer per metre length of tube. Use the following relation suggested by Nusselt.**

$$\mathbf{Nu = C(R_{ed})^n \left(\frac{T_w}{T_a}\right)^{0.25\,n}}$$

**Where \[Tw\] and \[Ta\] are the absolute temperatures of the tube surface and air.**

**Take C=0.6 and n = 0.466 if 40<Red <4000**

**Take the following properties of air at** $\left(\frac{90+650}{2}\right) = 370°C$

$$\mathbf{K = 49.13\times 10^{-3}\,W/m-k}$$

$$\mathbf{\nu = 57.5 \times 10^{-6}\ m^2/s.}$$

**Solution:**

$$R_{ed} = \frac{Vd}{\nu} = \frac{8\times 0.025}{57.5\times 10^{-6}} = 348C$$

$$\frac{T_w}{T_a} = \left(\frac{650+273}{90+273}\right) = 2.54$$

$$\therefore\ Nu = \frac{hd}{K} = 0.6(3480)^{0.466}\ (2.54)^{0.26\times 0.466} = 30$$

$$h = \frac{30\ K}{d} = \frac{30\times 49.13\times 10^{-3}}{0.025} = 58.8\ W/m^2K$$

The rate of heat transfer per metre length of tube

$$Q = hA(T_w - T_a)$$

$$=58.8\times(\pi\times 0\ 025\times 1)(650-90)$$

$$=2590W=2.59\ kW=2\ 95\ kJ/s.$$

**Problem 11.2 Calculate the heat transfer coefficient for water flowing through a 2 cm diameter tube with a velocity of 2.5 m/s. The average temperature of the water is 50°C and surface temperature of the tube is slightly below this temperature.**

**Assume the flow is turbulent.**

**The properties at 50°C are given below.**

**$C_p$ = 4180 J/kgK, K = 0.643 W/mK**

$\rho = 988 \text{ kg/m}^3$, $\mu = 544 \times 10^{-6} \text{kg/ms}$.

**Solution:**

As the flow is turbulent, we can use

$Nu_\alpha = 0.02\, Re^{0.8} Pr^{0.4}$

$$Re = \frac{\rho dV}{\mu} = \frac{988 \times 0.02 \times 2.5}{544 \times 10^{-6}} = 90800$$

$$Pr = \frac{\mu Cp}{K} = \frac{544 \times 10^{-6} \times 4182}{0.643} = 3054$$

$$\therefore Nu_\alpha = \frac{h_\alpha d}{K} = 0.023\,(90800)^{0.8}(3.54)^{0.4} = 353$$

$$\therefore h_\alpha = \frac{353}{1} \times \frac{K}{d} = \frac{353 \times 0.643}{0.02} = 11350 \text{ W/m}^2 - \text{K}$$

$$= 11.35 \text{ kW/m}^2 - \text{K}.$$

**Problem 10.8 Estimate the heat transfer from a 40 W incandescent bulb at 125°C to 25°C in quiescent air. Approximate the bulb as a 50 mm diameter sphere. What percent of the power is lost by free convection?**

**The appropriate correlation for the convection for the convection coefficient is**

**Nu = 0.60 (Gr Pr)$^{0.25}$**

**Where the different parameters are evaluated at the mean film temperature, and the characteristic length is the diameter of the sphere.**

**Solution:** At the mean film temperature, $t_f$ = (125+25)/2=75°C, the thermo- physical properties for air are:

$\nu = 20.55 \times 10^{-6} m^2/s$ ; $k = 0.03 W/m - deg$

Pr = 0.693 and $\beta = \frac{1}{273+75} = 2.87 \times 10^{-3}$ per deg kelvin

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} = \frac{l^3 \rho^2 \beta g \Delta t}{\nu^2}$$

$$= \frac{(0.05)^3 \times 2.87 \times 10^{-3} \times 9.81 \times (125-25)}{(20.55 \times 10^{-6})^2} = 8.0 \times 10^5$$

Using the given correlation,

$$Nu = \frac{hl}{k} = 0.60(8.0 \times 10^5 \times 0.693)^{0.25}$$

$$h = \frac{0.03}{0.05} \times 0.60(8.0 \times 10^5 \times 0.693)^{0.25} = 9.823\ W/m^2K$$

This gives a heat transfer of

$$Q = h\,A\,\Delta t = 9.823 \times \pi \times 0.05^2 \times (125 - 25) = 7.71\ W$$

Therefore the percent of heat loss by free convection is:

$$\frac{7.71}{40} \times 100 = 19.278\%$$

**Problem 10.9 A hot square plate 40 cm 40 cm at 100°C is exposed to atmospheric air at 20°C. Make calculations for the heat loss from both surfaces of the plate, if a the plate is kept vertical (b) plate is kept horizontal.**

**The following empirical aorrelations have been suggested:**

**Nu = 0.125 (Gr Pr)$^{0.33}$ for vertical position of plate, and**

**Nu = 0.72 (Gr Pr)$^{0.25}$ for upper surface**

**= 0.35 (Gr Pr)$^{0.25}$ for lower surface**

**Where the air properties are evaluated at the mean temperature.**

**Solution:**

At the mean temperature, $t = (100 + 20)/2 = 60°C$, the thermo physical properties of air are

$$\rho = 1.06\ kg/m^3, k = 0.028\ W/m - deg$$

$$C_p = 1.008\ kj/kg\ \ K\ \ and\ \ \nu = 18.97 \times 10^{-6}\ m^2/s$$

$$\beta = \frac{1}{T} = \frac{1}{273+60} = 0.003\ \text{per degree kelvin}$$

$$\therefore \quad Pr = \frac{\mu c_p}{k} = \frac{\rho \nu c_p}{k} = \frac{1.06 \times (18.97 \times 10^{-6}) \times (1.008 \times 1000)}{0.028} = 0.724$$

$$Gr = \frac{l^3 \rho^2 \beta g \Delta t}{\mu^2} = \frac{l^3 \beta g \Delta t}{\nu^2}$$

$$= \frac{0.4^3 \times 0.003 \times 9.81(100-200)}{(18.97 \times 10^{-6})^2} = 4.19 \times 10^8$$

$$Gr \times pr = (4.19 \times 10^8) \times 0.724 = 3.033 \times 10^8$$

(a) When the plate is oriented vertically,

$$Nu = 0.125 \times (3.033 \times 10^8)^{0.33} = 78.69$$

$$h = Nu \times \frac{k}{l} = 78.69 \times \frac{0.028}{0.4} = 5.508\ W/m^2\ K$$

This gives a heat transfer of : Q = 2 h A Δt

The factor 2 accounts for two sides of the plate

$$Q = 2 \times 5.508 \times (0.4 \times 0.4) \times (100 - 20) = 141\ W$$

(b) When the plate is positioned horizontally

(i) For upper surface:

$$Nu = 0.72(3.003 \times 10^8)^{0.25} = 95$$

$$h = Nu \frac{k}{l} = 95 \times \frac{0.028}{0.4} = 6.65\ W/m^2\ K$$

$$Q_u = h\ A\ \Delta t = 6.65 \times (0.4 \times 0.4) \times (100 - 20) = 85.12\ W$$

(ii) For lower surface:

$$Nu = 0.35(3.003 \times 10^8)^{0.25} = 46.19$$

$$h = Nu\frac{k}{l} = 46.19 \times \frac{0.028}{0.4} = 3.23\ W/m^2\ K$$

$$Q_l = h\ A\ \Delta t = 3.23 \times (0.4 \times 0.4) \times (100 - 20) = 41.35\ W$$

$$\therefore \quad Q = Q_i + Q_l = 85.12 + 41.35 = 126.47\ W$$

**Comments:** The above calculations show that the plate loses more heat when it is oriented vertically. Obviously natural cooling can be achieved more effectively by keeping the plate in vertical position.

**Problem 11.3 Air at 25°C and 1 bar flows over a flat plate at a speed 1.25 m/s. Calculate the boundary layer thickness at distances of 15 cm and 30 cm from the leading edge of the plate. What would be the mass entrainment (mass flow entering the boundary layer) between these two sections? Assume parabolic velocity distribution**

$$\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

**The velocity of air at 25°C is stated to be 6.62kg/hr m.**

**Solution:**

The density of air is calculated from the characteristic gas equation: $p = \rho RT$

Given: $p = 1\text{bar} = 1 \times 10^5 \text{N/m}^2$

$T = 25°\text{C} = (25 + 273) = 298 \text{ K}$

$R = 287 \text{ J/kg K}$

$$\therefore \quad \rho = \frac{p}{RT} = \frac{1 \times 10^5}{287 \times 298} = 1.169 \text{ kg/m}^3$$

The flow Reynolds number is, $Re_x = (x \rho U_\infty)/\mu$

At $x = 15$ cm ; $Re_x = \frac{0.15 \times 1.169 \times (1.25 \times 3600)}{6.62 \times 10^{-2}} = 11919$

At $x = 30$ cm ; $Re_x = \frac{0.3 \times 1.169 \times (1.25 \times 3600)}{6.62 \times 10^{-2}} = 23838$

For the given parabolic velocity distribution, the boundary layer thickness is prescribed by the relation,

$$\delta = \frac{4.64}{\sqrt{Re_x}}$$

$\therefore$ At $x = 15$ cm ; $\delta_1 = \frac{4.64 \times 0.15}{\sqrt{11919}} = 6.37 \times 10^{-3}$ m

$\therefore$ At $x = 30$ cm ; $\delta_2 = \frac{4.64 \times 0.30}{\sqrt{23838}} = 8.99 \times 10^{-3}$ m

(b) At any position, the mass flow in the boundary layer is given by the integral

$$m_x = \int_0^\delta \rho\, u\, dy$$

Where the velocity is given by

$$u = U_\infty \left\{\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}$$

Evaluating the integral with this velocity distribution,

$$m_x = \int_0^\delta \rho \left[U_\infty \left\{\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right\}\right] dy$$

$$\rho U_\infty \left[\frac{3}{4}\frac{y^2}{\delta} - \frac{1}{8}\frac{y^4}{\delta^3}\right]_0^\delta = \frac{5}{8}\rho U_\infty \delta$$

Thus the mass entrainment between the two sections is:

$$\delta_m = \frac{5}{8}\rho U_\infty (\delta_2 - \delta_1)$$

$$= \frac{5}{8} 1.168 \times 1.25(8.99 \times 10^{-3} - 6.37 \times 10^{-3})$$

$$= 2.393 \times 10^{-3}\ \text{kg/s} = 8.61\ \text{kg/hr}$$

**Example11.4 A small thermo-couple is positioned in a thermal boundary layer near a flat plate past which water flows at 30°C and 0.15 m/s. The plate is heated to a surface temperature of 50°C and at the location of the proble, the thickness of thermal boundary layer is 15mm. if the temperature profile as measured by the probe is well-represented by**

$$\frac{t - t_s}{t_\infty - t_s} = 1.5\left(\frac{y}{\delta_t}\right) - 0.5\left(\frac{y}{\delta_t}\right)^3$$

**Determine (a) the heat flux from plate to water; and (b) the heat transfer coefficient**

**Solution:** At the mean film temperature $t_f$ = (30+50)/2=40°C, the thermal conductivity of water is 0.633 W/m-deg.

$$\frac{Q}{A} = -k(t_\infty - t_s)\frac{\partial}{\partial y}\left(\frac{t - t_s}{t_\infty - t_s}\right)_{y=0}$$

$$= -k(t_\infty - t_s)\left|\frac{1.5}{\delta_t} - \frac{3 \times 0.5}{\delta_t} y^2\right|_{y=0} = \frac{1.5k(t_s - t_\infty)}{\delta_t}$$

$$= \frac{1.5 \times 0.633 \times (50 - 30)}{0.015} = 1266\ \text{W/m}^2$$

(b) Heat transfer coefficient,

$$h = \frac{Q/A}{(t_s - t_\infty)}$$

$$= \frac{1.5k(t_s - t_\infty)}{\delta_t} \times \frac{1}{t_s - t_\infty}$$

$$= \frac{1.5k}{\delta_t} = \frac{1.5 \times 0.633}{0.05} = \mathbf{63.3\ w/m^2 - deg}$$

**Example11.5 Air at 25°C approaches a 0.9m long by 0.6 m wide flat plate with an approach velocity 4.5 m/s. The plate is heated to a surface temperature of 135°C. Make calculations for:**

**a) Local heat transfer coefficient at a distance of 0.5 m from the leading edge;**

**b) Total rate of heat transfer from the plate to the air**

**Solution:** At the mean film temperature $t_f$ = (25+135)/2=80°C, the thermo-physical properties of air are:

$v = 21.0910^{-6}\ m^2/s$ ; $k = 0.0304$ W/m-deg ; $p_r = 0.692$

a) $Re_x = \frac{U_\infty x}{V} = \frac{4.5 \times 0.5}{21.09 \times 10^{-6}} = 1.067 \times 10^5 < 5 \times 10^5$

The flow is laminar at a distance of 0.5 m from the leading edge, and accordingly

$$N_{ux} = 0.332(Re_x)^{0.5}(Pr)^{0.33} = 0.332(1.067 \times 10^5)^{0.5} \times (0.692)^{0.33} = 95.98$$

$$\therefore h_x = N_{ux} \times \frac{k}{x} = 95.98 \times \frac{0.0304}{0.5} = 5.83\ W/m^2K$$

b) For the entire plate length l=0.9 m

$$Re_x = \frac{U_\infty x}{V} = \frac{4.5 \times 0.5}{21.09 \times 10^{-6}} = 1.923 \times 10^5 < 5 \times 10^5$$

Obviously the film is laminar for the entire plate length, and accordingly

$$\bar{N}_u = 0.664(R_{el})^{0.5}(Pr)^{0.33}$$

$$= 0.664(1.923 \times 10^5) \times (0.692)^{0.33} = 257.69$$

$$\bar{h} = \bar{N}_u \times \frac{k}{l} = 257.69 \times \frac{0.0304}{0.9} = 8.70\ \text{W/m-deg}$$

Heat loss from one side of the plate,

$$Q = hA\,\Delta t = 8.70 \times (0.9 \times 0.6) \times (135 - 25) = \mathbf{516.78\ W}$$

**Example11.6 Atmospheric air at 30°C temperature and free stream velocity of 2.5 m/s flows along the length of plate maintained at a uniform surface temperature of 90°C. The length, width and thickness of the plate is 100 cm, 50 cm and 2.5 cm. if thermal conductivity of the plate material is 25 W/m-deg, make calculations for (a) heat lost by the plate; (b) temperature of bottom surface of the plate for steady state conditions.**

**Solution:** At the mean film temperature $t_f$ = (50+100)/2=75°C, the thermo-physical properties of air are:

$$\rho = 1.06\ \text{kg/m}^3\ ;\ c = 1005\ \text{j/kg K}\ ;\ \Pr = 0.696$$

$$k = 0.0289\ \text{W/m-deg and}\ \mu = 20 \times 10^{-5}\ \text{kg/ms(Ns/m}^2)$$

$$Re_1 = \frac{\rho V l}{\mu} = \frac{1.06 \times 2 \times 1}{20 \times 10^{-6}} = 1.06 \times 10^5$$

The Reynolds number is less than $510^5$, hence the flow is laminar and accordingly

$$\bar{N}_u = \frac{\bar{h} l}{k} = 0.664 (Re_1)^{0.5} (Pr)^{0.33}$$

$$= 0.664\ (1.325 \times 10^5)^{0.5} \times (0.696)^{0.33} = 214.39$$

And $$\bar{h} = \bar{N}_u \times \frac{k}{l} = 214.39 \times \frac{0.0289}{1} = 6.196\ \text{W/m}^2\text{-deg}$$

Convective heat loss from the plate,

$$Q = \bar{h}\ A\ \Delta t = 6.196 \times (1.0 \times 0.5) \times (100 - 50) = \mathbf{154.9W}$$

This convective heat loss must be conducted through the plate. Then from energy balance

$$Q = -\frac{k A (t_s - t_b)}{\delta}\ \ ;\ \ 154.9 = -\frac{25 \times (1.0 \times 0.5) \times (100 - t_b)}{0.025}$$

Bottom temperature of the plate,

$$t_b = 100 + \frac{0.025 \times 154.9}{25 \times (1.0 \times 0.5)} = \mathbf{103.098°C}$$

## Lesson 16. Laminar Forced Flow in a Tube

Consider a circular tube of radius ‘R’ and length ‘L’ through which a incompressible fluid is flowing. In this tube, a concentric cylinder of radius ‘r’, length ‘dx’ located at a distance ‘y’ from the bottom of the tube is considered as shown in Figure 2.

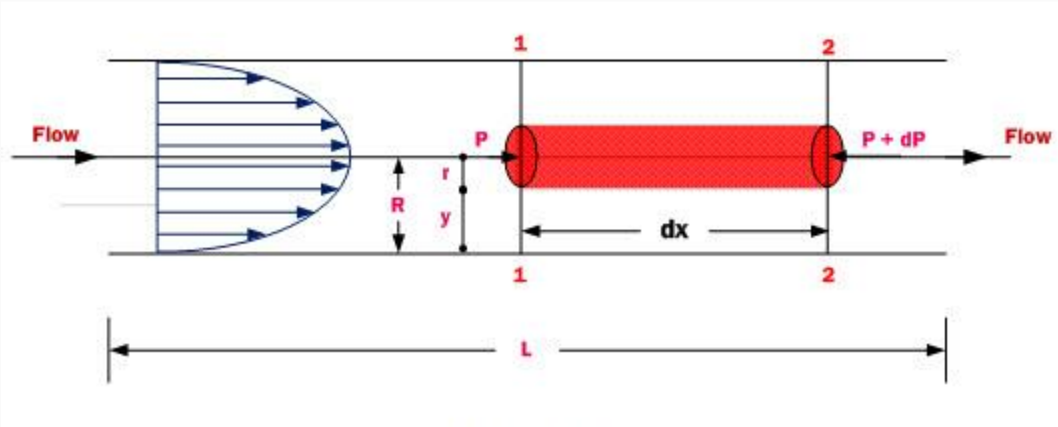



Figure 2

For a fully developed flow, velocity at sections 1-1 and 2-2 of the concentric cylinder is equal and pressure forces are balanced by the viscous shear forces.

Net pressure force acting on concentric cylinder

$$= P\pi {r^2} - \left( {P + \frac{{\partial P}}{{\partial x}}dx} \right)\pi {r^2} \qquad (1)$$

Viscous Shear force acting

$$= 2\pi rdx\, \times \,\tau \qquad (2)$$

Under equilibrium, net pressure force is equal to viscous shear force

$$\left( { - \frac{{\partial P}}{{\partial x}}dx} \right)\pi {r^2} = 2\pi rdx\, \times \,\tau \qquad (3)$$

Dividing both sides of the equation (3) by volume of concentric cylinder, $\pi {r^2}dx$ , we get

$$\frac{{\partial P}}{{\partial x}}=- \frac{{2\tau }}{r}$$

Or $$\frac{{dP}}{{dx}}=- \frac{{2\tau }}{r}$$

Or $$\tau=-\frac{r}{2}\frac{{dP}}{{dx}} \qquad (4)$$

Invoking Newton’s Law of viscosity for laminar flow

$$\tau= \mu \frac{{du}}{{dy}}=- \mu \frac{{du}}{{dr}} \qquad (5)$$

-ve sign is because ‘r’ is measured in a direction opposite to ‘y’.

Substituting the value of from equation (4) in equation (5), we get

$$- \frac{r}{2}\frac{{dP}}{{dx}}=- \mu \frac{{du}}{{dr}}$$

$$\frac{{du}}{{dr}} = \frac{r}{{2\mu }}\frac{{dP}}{{dx}} \qquad (6)$$

Integrating equation (6) twice wrt to 'r', we get

$$u = \frac{{{r^2}}}{{4\mu }}\frac{{dP}}{{dx}} + C \qquad (7)$$

Applying boundary condition to equation (7)

At $r=R$, $u=0$

$$C =- \frac{1}{{4\mu }}\frac{{dP}}{{dx}}{R^2} \qquad (8)$$

Substituting the value of constant C in equation (7), we get

$$u = \frac{1}{{4\mu }}\left( { - \frac{{dP}}{{dx}}} \right)\left( {{R^2} - {r^2}} \right) \qquad (9)$$

Equation (9) gives velocity distribution and location of maximum velocity can be obtained by differentiating equation (9) with respect to 'x' and equate it equal to zero.

$$\frac{{du}}{{dx}} = \frac{1}{{4\mu }}\left( { - \frac{{dP}}{{dx}}} \right)\left( { - 2r} \right) = 0$$

Or $r=0$

Therefore, maximum velocity occurs at centre of pipe and its value is given by

$${u_{\max }} = \frac{1}{{4\mu }}\left( { - \frac{{dP}}{{dx}}} \right)\left( {{R^2}} \right) \qquad (10)$$

Dividing equation (9) by equation (10), we get

$$\frac{u}{{{u_{\max }}}} = 1 - {\left( {\frac{r}{R}} \right)^2} \qquad (11)$$

**Average Velocity:**

In order to determine the average velocity, volumetric flow is equated to the integrated paraboloid flow.

$$V\,\pi {R^2} = \int\limits_0^R {u\,(2\pi r)dr} \qquad (12)$$

Substituting the value of 'u' from equation (11) in equation (12), we get

$$V\,\pi {R^2} = \int\limits_0^R {{u_{\max }}\,(1 - {{\left( {\frac{r}{R}} \right)}^2})2\pi rdr}$$

$$V\,\pi {R^2} = 2\pi {u_{\max }}\int\limits_0^R {\,(r - \frac{{{r^3}}}{{{R^2}}})dr}$$

$$V\,\pi {R^2} = 2\pi {u_{\max }}\left| {\frac{{{r^2}}}{2} - \frac{{{r^4}}}{{4{R^2}}}} \right|_0^R$$

$$V\,\pi {R^2} = 2\pi {u_{\max }}\left| {\frac{{{R^2}}}{2} - \frac{{{R^4}}}{{4{R^2}}}} \right|\, = \,2\pi {u_{\max }}\frac{{{R^2}}}{4}$$

$$V\, = \,\frac{{{u_{\max }}}}{2} \qquad (13)$$

Substituting value of '$u_{max}$' from equation (10) in equation (43), we get

$$V\, = \,\frac{1}{{8\mu }}\left( { - \frac{{dP}}{{dx}}} \right)\left( {{R^2}} \right) \qquad (14)$$

The pressure gradient dP/dx is usually is expressed in terms of friction factor and is given as

$$- \frac{{dP}}{{dx}}\, = \,\frac{f}{d}\frac{{\rho {V^2}}}{2} \qquad (15)$$

Where $\frac{{\rho {V^2}}}{2}$ is the dynamic pressure of mean flow

d is the tube diameter

f is the friction factor

From equations (14) and (15), we get

$$f = \frac{64}{\rho V d / \mu} = \frac{64}{\mathrm{Re}} \qquad (16)$$

The above equation is valid for laminar flow, Re<2000

Using equation (15), pressure drop for a finite length between points $x_1$ and $x_2$, which are 'L' distance apart, can be expressed as

$$\frac{{{P_1} - {P_2}}}{{{x_2} - {x_1}}}\, = \,\frac{f}{d}\frac{{\rho {V^2}}}{2}$$

$${P_1} - {P_2} = \frac{f}{d}\frac{{\rho {V^2}}}{2}\left( {{x_2} - {x_1}} \right)$$

$${P_1} - {P_2} = \frac{f}{d}\frac{{\rho {V^2}}}{2}\left( L \right) \qquad (17)$$

Substituting the value of 'f' from equation (16) in equation (17), we get

$$P_1 - P_2 = \frac{64}{\left( \rho V d / \mu \right)} \frac{\rho V^2}{2d} (L)$$

$$\frac{P_1 - P_2}{\rho} = \frac{32 \mu V L}{\rho d^2} = \frac{128 \mu Q L}{\rho \pi d^4} \qquad (18)$$

Where 'Q' is volumetric flow through the tube and is expressed as

Q= Velocity X Area of cross-section of the tube

 = V X (π/4) $d^2$

Equation (18) is known as Hagen-Poiseuille equation and represents a fully developed flow in which velocity profile dose not vary along the pipe axis.

**Temperature Distribution:**

In order to develop temperature distribution, let us consider flow of heat through an elementary area of length dx and thickness dr as shown in Figure 1 in which conduction along the axis is neglected.

Heat being conducted in to the element is expressed as

$$d{Q_r} =- k2\pi rdx\frac{{\partial T}}{{\partial r}} \qquad (19)$$

Heat being conducted out of the element is expressed as

$$d{Q_{r + dr}} =- k2\pi \left( {r + dr} \right)dx\frac{\partial }{{\partial r}}\left( {T + \frac{{\partial T}}{{\partial r}}dr} \right) \qquad (20)$$

Net heat conducted in to the element is obtained by subtracting equation (19) from equation (20) and is equal to

$$- k2\pi rdx\frac{{\partial T}}{{\partial r}}+- k2\pi \left( {r + dr} \right)dx\left( {\frac{{\partial T}}{{\partial r}} + \frac{{{\partial ^2}T}}{{\partial {r^2}}}dr} \right) \qquad (21)$$

Net heat convected out of the element is given as

$$d{Q_{conv}}=\rho\left( {2\pi rdr} \right)u{C_p}\frac{{\partial T}}{{\partial x}}dx \qquad (22)$$

Under equilibrium conditions, net heat conducted in to the element is equal to the net heat convected out

$$- k2\pi rdx\frac{{\partial T}}{{\partial r}}+ - k2\pi \left( {r + dr} \right)dx\left( {\frac{{\partial T}}{{\partial r}} + \frac{{{\partial ^2}T}}{{\partial {r^2}}}dr} \right) = \rho \left( {2\pi rdr} \right)u{C_p}\frac{{\partial T}}{{\partial x}}dx \qquad (23)$$

Simplification and re-arrangement of equation (23) yields

$$\frac{1}{{ur}}\frac{\partial }{{\partial r}}\left( {r\frac{{\partial T}}{{\partial r}}} \right) = \frac{1}{\alpha }\frac{{\partial T}}{{\partial x}} \qquad (24)$$

Substituting the value of velocity ‘u’ from equation (11) in equation (24), we get

$$\frac{\partial }{{\partial r}}\left( {r\frac{{\partial T}}{{\partial r}}} \right)=\frac{1}{\alpha }\frac{{\partial T}}{{\partial x}}{u_{\max }}\left( {1 - {{\left( {\frac{r}{R}} \right)}^2}} \right)r \qquad (25)$$

Integrating equation (25) twice with respect to ‘r’, we get

$$\left( {r\frac{{\partial T}}{{\partial r}}} \right)=\frac{1}{\alpha }\frac{{\partial T}}{{\partial x}}{u_{\max }}\left( {\frac{{{r^2}}}{2} - \frac{{{r^4}}}{{4{R^2}}}} \right) + {C_1}$$

$$\left(\frac{\partial T}{\partial r}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial x}u_{\max}\left(\frac{r}{2} - \frac{r^3}{4R^2}\right) + C_1/r \qquad (26)$$

$$T = \frac{1}{\alpha }\frac{{\partial T}}{{\partial x}}{u_{\max }}\left( {\frac{{{r^2}}}{4} - \frac{{{r^4}}}{{16{R^2}}}} \right) + {C_1}{\log _e}r + {C_2} \qquad (27)$$

Applying the boundary conditions,

At r=0, $\left( {\frac{{\partial T}}{{\partial r}}} \right) = 0$ and again at r=0, $T=T_c$, we get

$C_1=0$ and $C_2=T_c$

Substituting the values of $C_1$ and $C_2$ in equation (27), we get

$$T = \frac{1}{\alpha }\frac{{\partial T}}{{\partial x}}{u_{\max }}\left( {\frac{{{r^2}}}{4} - \frac{{{r^4}}}{{16{R^2}}}} \right) + {T_c} \qquad (28)$$

Multiplying and dividing the right side of equation (28) by $R^2$, we get

$$T = \frac{{{u_{\max }}{R^2}}}{\alpha }\frac{{\partial T}}{{\partial x}}\left( {\frac{{{r^2}}}{{4{R^2}}} - \frac{{{r^4}}}{{16{R^4}}}} \right) + {T_c}$$

$$T - {T_c} = \frac{{{u_{\max }}{R^2}}}{{4\alpha }}\frac{{\partial T}}{{\partial x}}\left( {{{\left( {\frac{r}{R}} \right)}^2} - \frac{1}{4}{{\left( {\frac{r}{R}} \right)}^4}} \right) \qquad (29)$$

Equation (29) represents the temperature distribution.

By using fundamental heat conduction and convection equations, heat flux coefficient can be obtained.

$$Q = hA({T_w} - {T_b}) =- kA{\left( {\frac{{\partial T}}{{\partial r}}} \right)_{r = 0}} \qquad (30)$$

Where $T_w$ is the wall temperature and $T_b$ is the average bulk temperature of the fluid and mathematically it is expressed as

$${T_b} = \frac{{\int\limits_0^R {\rho (2\pi rdr)u{C_p}T} }}{{\int\limits_0^R {\rho (2\pi rdr)u{C_p}} }} \qquad (31)$$

For an incompressible fluid having constant density and specific heat, equation (31) can be expressed as

$${T_b} = \frac{{\int\limits_0^R {uTrdr} }}{{\int\limits_0^R {urdr} }} \qquad (32)$$

Generally, average value of bulk fluid temperature is used in the calculation of average heat transfer coefficient and given as

$${T_b} = \frac{{{T_{inlet}} + {T_{outlet}}}}{2}$$

**Combined Natural and Forced Convection**

Heat transfer by Natural convection occurs in a fluid if there exists a temperature gradient with in the fluid body. Heat transfer by forced convection is always accompanied by natural convection. Since heat transfer coefficient in case of forced convection are much higher as compared to the heat transfer coefficients associated with natural convection, therefore, heat transfer by natural convection is generally neglected in case of forced convection. If velocity of fluid is high, effect of neglecting natural convection will not introduce any appreciable error, however at low velocities, the effect of ignoring natural convection may result in considerable error. Therefore, it becomes imperative to define a criterion to assess the magnitude of natural convection in forced convection.

The parameter $Gr/Re^2$ represents relative significance of natural convection to forced convection as heat transfer coefficient is function of Grashoff number, Gr in natural convection and Reynolds number, Re in forced convection.

If $Gr/Re^2 < 0.1$, Natural convection is neglected.

If $Gr/Re^2 > 10$, Forced convection is neglected.

If $0.1 < Gr/Re^2 < 10$, Neither Natural nor forced convection is negligible. Both are to be considered.

In case when both natural and forced convection are considered, natural convection may enhance or reduce the natural convection depending upon the relative motion of buoyancy induced and forced convection currents.

In assisting flow, convectional currents set up due of density differences (natural convection) move in the same direction as the forced motion and natural convection assists the forced convection resulting in enhanced the heat transfer.

In opposing flow, convectional currents set up due of density differences (natural convection) move in the direction opposite to the forced motion and natural convection resists the forced convection resulting in decrease in heat transfer.

In transverse flow, convectional currents set up due of density differences (natural convection) move in a direction perpendicular to forced motion and results in mixing which enhances the heat transfer.

In order to determine heat transfer under combined natural and forced convection, following correlation is used.

$$Nu_{Combined} = \left( Nu^n_{forced} \pm Nu^n_{Natural} \right)^{1/n} \qquad (33)$$

$Nu_{forced}$ and $Nu_{Natural}$ are determined from the correlations for pure forced and pure natural convection correlations. The plus sign is for assisting and transverse flow and negative sign is for opposing flows. Value of 'n' varies between 3 and 4.

**Module 3. Radiation**

## Lesson 18. Radiation

As discussed earlier in Lesson 1, heat is transferred between two bodies even though they are not in direct physical contact and vacuum exists between these bodies. Such a heat transfer is termed as radiation heat transfer that involves emission of energy by heat exchanging bodies due to changes in electronic configurations of their molecules or atoms. Energy is emitted in the form of electromagnetic radiation. The electromagnetic radiation that can be detected as heat is termed as thermal radiation. Thermal radiation emitted by a body or substance is directly proportional to its temperature. Thermal Radiation is one of the modes of heat transfer and it differs from conduction and convection modes of heat transfer in following aspects:

i) Conductive and convective heat transfer depends upon temperature difference (raised to power one) between heat exchanging bodies whereas in radiation mode, heat transfer is directly proportional to the difference of the fourth power of absolute temperatures of individual bodies.

ii) Presence of an intervening medium between two bodies, exchanging heat by radiation mode, is not necessary. Rather, heat transfer by radiation mode takes place most efficiently in vacuum. In case of heat transfer by conduction and convection modes, presence of intervening medium is a must for exchange of heat between two bodies.

iii) Heat transfer by conduction and convection takes place from high temperature body to low temperature body. In radiation mode of heat transfer, thermal energy is not only emitted by high temperature body towards low temperature body but also from low temperature body towards high temperature body. However, net transfer of energy is always from high to low temperature body.

**Basic Theory of Radiation Heat Transfer:**

Transfer of heat by from the Sun to the earth by thermal radiations is the prime source of energy and without which survival of mankind can not be imagined. Two theories have been postulated to explain the phenomenon of heat transfer by radiation.

**i) Electromagnetic Wave Theory**

**ii) Quantum Theory**

**i) Electromagnetic Wave Theory**

The electromagnetic wave theory of radiation was first postulated by James Clerk Maxwell in 1864. According to this theory, heat transfer by radiation mode takes place as thermal energy is emitted by a body in the form of electromagnetic waves due to change in electronic configuration of its molecules or atoms. The emitted electromagnetic waves travel through space that is assumed to be filled with an hypothetical medium called ether and are absorbed by another body.he absorbing body reconverts these electromagnetic waves in to thermal energy. Re-conversion of energy carried by electromagnetic waves into thermal energy depends upon the material and surface characteristics of the absorbing body. Internal energy of emitting body decreases as reflected by decrease in its temperature with a corresponding increase in temperature of the absorbing body resulting in increase in its internal energy.

**ii) Quantum Theory**

Max Planck postulated quantum theory of radiation in 1900. According to this theory, molecules and atoms of a hot body are in excited state due to high energy levels. These high energy level molecules and atoms tend to return to low energy levels and during this process, the body emits energy in the form of electromagnetic radiations. The energy is not emitted continuously but radiated in form of successive and separated quantities which are termed as quanta or photons. The size of each quantum is different and energy of quantum is expressed as

$$E = h\,v \qquad (1)$$

Where h is Planck's constant and is equal to $6.62 \times 10^{-34}$ J-sec

v is frequency of vibrations and is directly proportional to temperature of body

**Spectrum of Electromagnetic Radiation:**

In nature, energy is transferred from one place to another place by electromagnetic radiation. Electromagnetic radiation travel through space and has electrical and magnetic effects. Electromagnetic radiation of different types exists and one type differs from the other on account of its frequency and wavelength. The spectrum measures range of frequencies of all electromagnetic radiation and it includes

- **Radio**
- **Microwaves**
- **Infrared**
- **Visible**
- **Ultra-violet**
- **X-rays**
- **Gamma rays**

Electromagnetic spectrum has been shown in Figure1

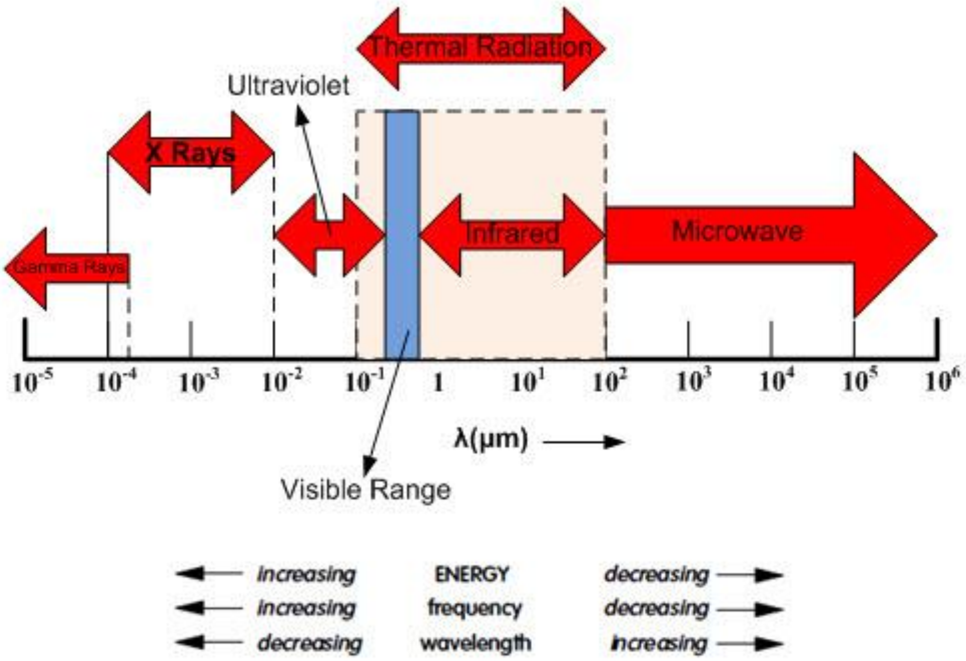



**Figure 1**

Every substance having temperature above absolute zero temperature emits electromagnetic radiation which is termed as thermal radiation. The rate of thermal radiation emission increases with increase in temperature of a substance. Range of wavelength of thermal radiation is between 0.1 to 100 μm and includes entire visible and infrared (IR) radiation as well as a portion of ultraviolet (UV) radiation. Following properties of thermal radiations are important from heat transfer point of view:

i) Thermal radiation travel through space in straight lines and do not heat up the space unless they are obstructed in their path.

ii) Nature and behavior of thermal radiation is same as that of visible light and they differ in wavelength only.

iii) Similar to visible light, thermal radiation obeys inverse square law and are also reflected as well as refracted.

**Reflection, Absorption and Transmission of Radiation:**

Thermal radiation is emitted by all substances and magnitude of emitted thermal radiation depends upon temperature of the emitting substance. Thermal radiation received by a body is partly absorbed, partly reflected and partly transmitted through the body. It has been shown in Figure 2 that Q amount of thermal radiation is incident upon a body, out of which $Q_t$ amount is transmitted through the body, $Q_r$ is reflected by the body and $Q_a$ is absorbed by the body. Therefore,

$$Q_t + Q_r + Q_a = Q \qquad (2)$$

Dividing both sides of equation (2) by Q, we get

$$\frac{Q_t}{Q} + \frac{Q_r}{Q} + \frac{Q_a}{Q} = \frac{Q}{Q} \qquad (3)$$

$$\tau + r + \alpha = 1$$

Where $\tau$ is transmissivity of the body and is equal to $\frac{Q_t}{Q}$

r is reflectivity of the body and is equal to $\frac{Q_r}{Q}$

$\alpha$ is absorptivity of the body and is equal to $\frac{Q_a}{Q}$

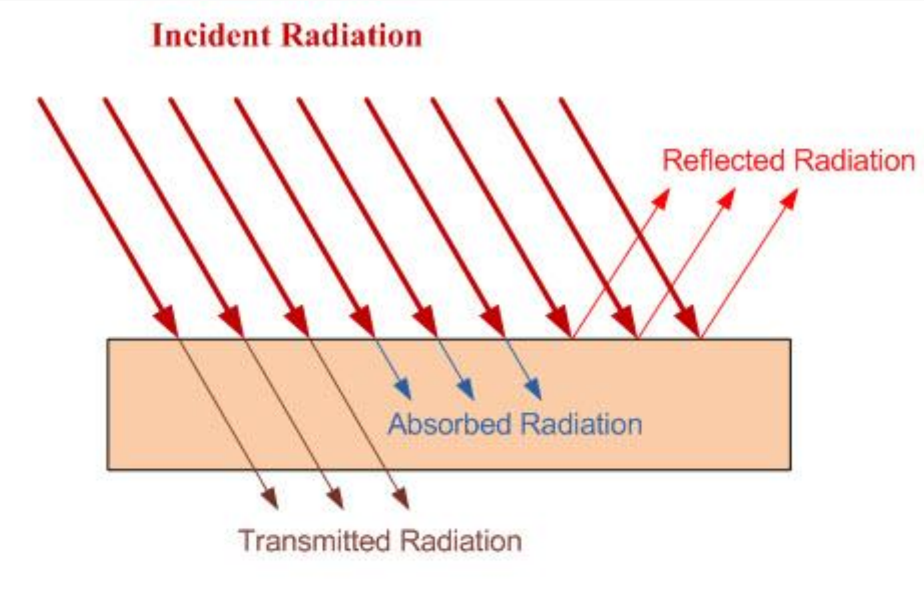



**Figure 2**

Absorptivity, transmissivity and reflectivity are dimensionless properties of the receiving body and values of these vary from 0 to 1.

Transmissivity of most of the solids and liquids is zero; however, there are few exceptions such as glass which allows transmittance of thermal radiation through it.

Therefore, for solids and liquids,

$$\alpha + r = 1$$

Depending upon values of absorptivity, reflectivity and transmissivity, bodies are classified as

| Type | Absorptivity, $\alpha$ | Transmissivity, $\tau$ | Reflectivity, r |
|---|---|---|---|
| Black Body | 1 | 0 | 0 |
| White Body | 0 | 0 | 1 |
| Transparent Body | 0 | 1 | 0 |
| Opaque Body* | Less than 1 | 0 | Less than 1 |
| For Most of Gases** | Less than 1 | Less than 1 | 0 |

* $\alpha + r = 1$

** $\alpha + \tau = 1$

**Reflection of Radiation:** Radiation reflection can be categorized as

- Specular Reflection: If angle of reflection is equal to angle of reflection, then it is called specular reflection as shown in Figure 3. Specular reflection occurs if surface of receiving body is highly polished.
- Diffused Reflection: If incident thermal radiation is reflected in all directions as shown in Figure 4, then it is called diffused reflection. Diffused reflection occurs if surface of receiving body is rough.

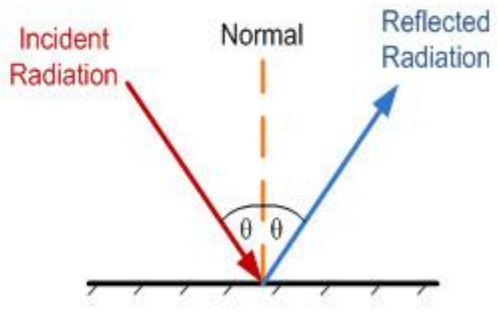



Figure 3

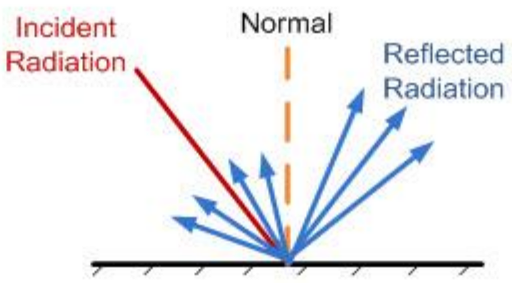



Figure 4

## Lesson 19. Emission of Radiation

**Emission of Radiation:**

As discussed earlier Lesson 16, energy emitted in form of electromagnetic radiation by bodies by virtue of their temperature is called thermal radiation. Thermal radiation emitted by a body is a function of temperature, condition of surface and material of emitting body. Therefore, two bodies maintained at same temperature emit different amounts of thermal radiation.

The ability of a body to emit radiation is called emission and magnitude of energy emitted per unit area per unit time by the body is called emissive power.

**Black Body:**

A perfect black body is one that absorbs all the thermal radiation, irrespective of wavelength, received by it. It does not reflect or transmit incident thermal radiation; therefore, absorptivity of such a body is 100%. At a given temperature and wavelength, a black body emits more energy as compared to any other body.

A perfect black body is a hypothetical body which behaves like a perfect absorber and emitter of thermal radiation and in practice no substance or body possesses properties of a perfect black body. However, bodies showing close approximation to a perfectly black body can be constructed.

Consider a hollow sphere with inside surface blackened and having a small hole at its surface. Thermal radiations entering the sphere through the hole are reflected repeatedly by the inner walls till they are completely absorbed. To avoid a direct reflection from inner surface, a pointed projection is made inside the sphere facing the hole as shown in Figure 1. Therefore, the small hole acts as a black body absorber.

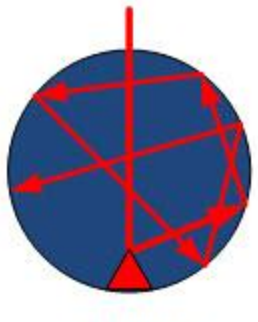


Figure 5

In order to compare radiative properties of real surfaces, a hypothetical body called blackbody is defined. A blackbody is an hypothetical body and has following properties.

- A black body absorbs all incident radiation irrespective of their wavelength and direction.
- At a given temperature and wavelength, energy emitted by a black body is the highest as compared to any other body.

- The radiation emitted by a black body depends upon wavelength and temperature, but it is independent of direction.

**Monochromatic emissive power ($E_{b\lambda}$) and Planck's Law:**

Monochromatic emissive power is defined as the energy emitted by a black body at a given wavelength in all directions per unit area per unit time and it is expressed in the units as W/($m^2$- μm). Using his quantum theory, Max Planck derived the expression for monochromatic emissive power ($E_{b\lambda}$) which is expressed as

$$\text{E}_{\text{b}\lambda} = \frac{2\pi c^2 h\,\lambda^{-5}}{\left(\exp\frac{ch}{\lambda kT}\right) - 1} \qquad (1)$$

Where

c – Velocity of light

= $2.98 \times 10^{8}$ m/sec

h – Planck's constant

= $6.625 \times 10^{-34}$ Joule-Sec

k – Boltzmann constant

= $1.3805 \times 10^{-23}$ Joule/K

λ – Wavelength monochromatic radiation emitted, m

T – Absolute temperature, K

Equation (1) can also be expressed as

$$\text{E}_{\text{b}\lambda} = \frac{C_1\,\lambda^{-5}}{\left(\exp\frac{C_2}{\lambda kT}\right) - 1} \qquad (2)$$

Where

$C_1 = 2\pi c^2 h = 37.404 \times 10^{-17}$ J-$m^2$/K

$C_2 = ch/k = 1.4387 \times 10^{-2}$ m – K

Equation (2) is known as Planck's law or Planck's distribution.

Monochromatic emissive power of a black body is a function of wave length and its variation with wavelength at selected temperatures has been plotted as shown in Figure 2. Following observations have made from this plot:

- At a specified temperature, monochromatic emissive power increases with increase in wavelength and attains a maximum value corresponding to a particular wavelength. However, further increase in wavelength results in decrease in emissive power.

- With increase in temperature, emissive power increases for all wavelengths.

- An increase in temperature results in decreases in value of wavelength for which emissive power is maximum.
- Area under each curve represents energy emitted by the black body at a particular temperature for the range of wavelength considered. With increase in temperature area under the curve increases as energy emitted increases.

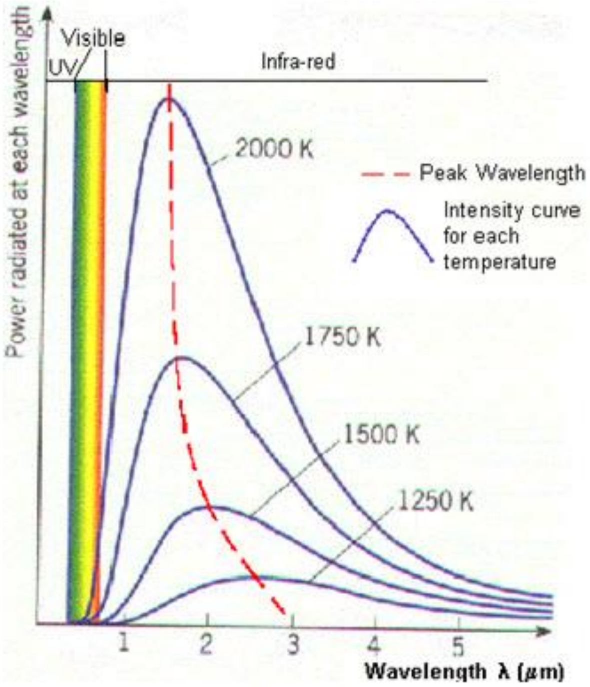



**Figure 2**

**Problem 7.7 A furnace inside temperature of 2250 K has a glass circular viewing of 6 cm diameter. If the transmissivity of glass is 0.08, make calculations for the heat loss from the glass window due to radiation.**

**Solution:** The radiation heat loss from the glass window is given by

$$Q = \sigma_b\, A\, T^4 \times \tau$$

Where \[\tau \] is the transmissivity of glass

$$Q = 5.67 \times 10^{-8} \times \frac{\pi}{4}(0.06)^2 \times 2250^4 \times 0.08 = 328.53 \text{ W}$$

**Problem 7.4 A thin metal plate of 4 cm diameter is suspended in atmospheric air whose temperature is 290 K. the plate attains a temperature of 295 K when one of its face receives radiant energy from a heat source at the rate of 2 W. If heat transfer coefficient on both surfaces of the plate is stated to be 87.5 W/m²-deg, workout the reflectivity of the plates.**

**Solution:** Heat lost by convection from both sides of the plate

$= 2h\, A\, \Delta t$

The factors 2 accounts for two sides of the plate

$$= 2 \times 87.5 \times \left\{\frac{\pi}{4}(0.04)^2\right\} \times (295 - 290) = 1.1 \text{ W}$$

For most of solids, the transmissivity is zero.

Energy lost by reflection = 2.0-1.1 =0.9 W

Reflectivity $\rho = \frac{Q_r}{Q} = \frac{0.9}{2.0} = 0.45.$

**Problem 7.9 A black body of total area 0.045 m² is completely enclosed in a space bounded by 5 cm thick walls. The walls have a surface area 0.5 m² and thermal conductivity 1.07 W/m-deg. If the inner surface of the enveloping wall is to be maintained at 215°C and the outer wall surface is at 30°C, calculate the temperature of the black body. Neglect the difference between inner and outer surfaces areas of enveloping material.**

**Solution:** Net heat radiated by the black body to the enclosing wall,

$$Q_r = \sigma_b A(T_b^4 - T_w^4)$$

$$= 5.67 \times 10^{-8} \times 0.045(T_b^4 - 488^4)$$

Where $T_b$ is the temperature of the black body in degree kelvin.

Heat conducted through the wall,

$$Q_c = \frac{kA\,\Delta t}{\delta} = \frac{1.07 \times 0.5 \times (215 - 30)}{0.05} = 1979.5\ W$$

Under steady state conditions, the heat conducted through the wall must equal the net radiation loss from the black body. Thus

$$= 5.67 \times 10^{-8} \times 0.045(T_b^4 - 488^4) = 1979.5$$

$$T_b^4 = \frac{1979.5}{5.67 \times 10^{-8} \times 0.045} + 488^4 = 8349.47 \times 10^8$$

Temperature of the black body, $T_b$ =955.9 K

**Problem 7.12 A furnace radiation at 2000K. Treating it as a black body radiation, calculate the**

**(i) Monochromatic radiant flux density at 1 µm wavelength.**

**(ii) Wavelength at which emission is maximum and the corresponding radiant flux density**

**(iii) Total emissive power, and**

**(iv) Wavelength λ such that emission from 0 to λ is equal to the emission from λ to ∞.**

**Solution:** (a) From Planck's law of distribution,

$$(E_\lambda)_b = \frac{C_1 \lambda^{-5}}{\exp\,[C_2/\lambda T]-1}$$

$$= \frac{0.374\times10^{-15}\times(1\times10^{-6})^{-5}}{\exp\,[1.4388\times10^{-2}/1\times10^{-6}\times2000]-1}$$

$$= \frac{0.374\times10^{15}}{1331.4-1} = \mathbf{2.81\times10^{7}W/m^2} \text{ per meter wavelength}$$

(b) From Wien's displacement law;

$$\therefore\ \lambda_{max} = \frac{2.898\times10^{-3}}{2000} = \mathbf{1.449\times10^{-6}m}$$

Maximum radiant flux density,

$$1.285\times10^{-5}T^5 = 1.285\times10^{-5}\times(2000)^5$$

$$= \mathbf{4.11\times10^{11}W/m^2}\text{per metre wavelength}$$

(c) From Stefan – Boltzman law,

$$E = \sigma_b T^4$$

$$= 5.67\times10^{-8}\times(2000)^4 = \mathbf{907200\ W/m^2}$$

(d) The emission in the band width 0 to λ is half of the total emission from 0 to ∞. Therefore, fractional emissive power is 0.5. Corresponding to $F_{0-\lambda}$ = 0.5, the wavelength temperature product as read from table 7.1 is approximately 4100 μm-K. Thus λT = 4100 and so

$$\lambda = \frac{4100}{2000} = 2.05\ \mu m$$

**Problem 7.16 A polished metal pipe 5 cm outside diameter and 370 K temperature at the outer surface is exposed to ambient conditions at 295 K temperature. The emissivity of the surface is 0.2 and the convection coefficient of heat transfer is 11.35 W/m²-deg. Calculate the heat transfer by radiation and natural convection per metre length of the pipe. Take thermal radiation constant $\sigma_b = 5.67\times10^{-8}W/m^2K^4$. What would be the overall coefficient of heat transfer by the combined mode of convection and radiation?**

**Solution:**

Surface area A of the pipe per metre run is equal to

$\pi dl = \pi \times 0.05 \times 1 = 0.157\ m^2$

$$Q_r(\text{radiation}) = \epsilon\ \sigma_b A\ (T_1^4 - T_2^4)$$

$$= 0.2 \times 5.67 \times 10^{-8} \times 0.157(370^4 - 295^4) = 19.88\ W$$

$$Q_c(\text{convection}) = h\ A\ \Delta t$$

$$= 11.35 \times 0.157(370 - 295)$$

$$= 133.64\ W$$

$$Q_t(\text{total}) = 19.88 + 133.64 = 153.52\ W$$

The total heat exchange can be expressed as:

$Q_t = U\ A\ \Delta T$ Where U is the overall coefficient of heat transfer

$$\therefore U = \frac{Q_t}{A\ \Delta T} = \frac{153.52}{0.157(370-295)} = 13.04\ W/m^2 - deg$$

**Problem 7.18 A gray surface has an emissivity at a temperature of 550 K source. If the surface is opaque, calculate its reflectivity for a black body radiation coming from a 550 K source.**

**(b) A small 25 mm square hole is made in the thin-walled door of a furnace whose inside walls are at 920 K. if the emissivity of the walls is 0.72, calculate the rate at which radiant energy escapes from the furnace through the hole to the room.**

**Solution:** The requirement that all of the radiant energy striking any surface may be accounted for is:

$$\alpha + \rho + \tau = 1$$

Here:

(i) $\tau = 0$ as the surface is opaque

(ii) $\alpha = \epsilon = 0.35$

This is in accordance with Kirchoff's law which states that absorptivity equals emissivity under the same temperature conditions.

$\therefore$ Reflectivity $\rho = 1 - (\alpha + \tau) = 1 - (0.35 + 0) = 0.65$

Thus the surface reflects 65 percent of incident energy coming from a source at 550 K.

(b) The small hole acts as a black body and accordingly the rate at which radiant energy leaves the hole is

$$E = \sigma_b A T^4$$

$$= 5.67 \times 10^{-8} \times (0.025 \times 0.025) \times 920^4$$

$$= \mathbf{25.38\ watts}$$

**Note:** The data about the emissivity of the inside wall is not needed.

## Lesson 20. Stefan-Boltzman Law

**Stefan-Boltzman Law:**

Total emissive power 'E' of a body is defined as total energy radiated by the body in all directions over entire range of wavelength per unit surface area per unit time. The total emissive power 'E' of a black body can be determined easily by using Stefan-Boltzman law if absolute temperature of the black body is known.

Using data of earlier experimental studies, Stefan discovered that total emissive power of a radiating body is directly proportional to the fourth power of absolute temperature of the body. Later on Boltzman theoretically proved the empirical equation proposed by Stefan on the basis of principles of thermodynamics. However, this proof was confined to black bodies.

The equation obtained on the basis of experimental work and proved theoretically was called as Stefan-Boltzman law which states that total energy emitted by a black body per unit area and per unit time is directly proportional to the fourth power of its absolute temperature and is expressed as:

$$E_b = \sigma_b T^4 \qquad (1)$$

$E_b$ - Energy emitted from a black body per unit area per unit time, $W/m^2$

$\sigma_b$ – Stefan-Boltzman constant

$= 5.678 \times 10^{-8}\ W/(m^2\text{-}K^4)$

T - Absolute temperature of the emitting surface, K.

Equation (1) represents energy emitted in the form of heat by a black body at a given temperature irrespective of heat that it has received from surroundings. A black body at temperature $T_1$, surrounded by another black body at temperature $T_2$ is emitting thermal radiation and at the same time receiving thermal radiation form the surrounding body, therefore, net energy per unit time lost by it is expressed as

$$E_{net} = \sigma_b\left(T_1^4\,-\,T_2^4\right) \qquad (2)$$

**Emissivity, ε**

Emissivity is a property of a body which is defined as the ratio of emissive power of the body to the emissive power of a hypothetical black body when both the bodies are maintained at same temperature.

$$\boldsymbol{\varepsilon} = \,\frac{E}{E_b} \qquad (3)$$

Depending upon the direction of emission of energy, emissivity can be categorized as

i) Monochromatic emissivity: It is the ratio of monochromatic emissive power of a body to the monochromatic emissive power of a black body at same wave length and temperature.

$$\boldsymbol{\varepsilon}_\lambda = \,\frac{\left(E_\lambda\right)}{\left(E_\lambda\right)_b} \qquad (4)$$

ii) Total emissivity: It is the ratio of the total emissive power of a body to the total emissive power of a black body maintained at same temperature.

$${\mathbf{\varepsilon }} = \,\frac{E}{{{E_b}}} \qquad (5)$$

iii) Normal total emissivity: It is the ratio of the normal component of total emissive power of a body to the normal component of total emissive power of a black body maintained at same temperature.

$${{\mathbf{\varepsilon }}_n} = \,\frac{{{E_n}}}{{{{\left( {{E_b}} \right)}_n}}} \qquad (6)$$

**Kirchoff's Law of Radiation**

According to Kirchoff's Law ratio of total emissive power to absorptivity is constant for all bodies which are in thermal equilibrium with their surroundings. For three bodies in thermal equilibrium with each other, it can be written

$$\frac{{{E_1}}}{{{\alpha _1}}} = \,\frac{{{E_2}}}{{{\alpha _2}}} = \frac{{{E_3}}}{{{\alpha _3}}} = Cons\tan t \qquad (7)$$

Assuming third body to be black body, equation (7) can be written as

$$\frac{{{E_1}}}{{{\alpha _1}}} = \,\frac{{{E_2}}}{{{\alpha _2}}} = \frac{{{E_b}}}{{{\alpha _b}}} = Cons\tan t \qquad (8)$$

Since for a black body, ${\alpha _b}$ =1, equation (8) can be written as

$$\frac{{{E_1}}}{{{\alpha _1}}} = \,\frac{{{E_2}}}{{{\alpha _2}}} = {E_b} \qquad (9)$$

$$\frac{{{E_1}}}{{{E_b}}} = \,{\alpha _1}\,,\,\,\,\,\,\frac{{{E_2}}}{{{E_b}}} = {\alpha _2} \qquad (10)$$

However, according to definition of emissivity,

$$\frac{{{E_1}}}{{{E_b}}} = \,{\varepsilon _1}\,,\,\,\,\,\,\frac{{{E_2}}}{{{E_b}}} = {\varepsilon _2} \qquad (11)$$

Comparing equations (10) and (11), we can write

$${\varepsilon _1} = \,{\alpha _1}\,,\,\,\,\,\,{\varepsilon _2} = {\alpha _2}$$

Or $$\varepsilon= \,\alpha \qquad (12)$$

Therefore, Kirchoff's law states that for a body in thermal equilibrium with its surroundings, its absorptivity is equal to its emissivity.

**Gray body and its emissive power:**

For a given temperature absorptivity and emissivity of a black surface are equal and it emits more energy as compared to a real surface. Emissivity and absorptivity of a real surface ma vary with temperature or wavelength of radiations emitted or absorbed. However for a gray body emissivity and absorptivity are independent of wavelength. A gray body is defined as a surface whose emissivity is constant at all temperatures and throughout the entire range of wavelength. Therefore, a gray body, like black body, is an ideal body and values of its absorptivity and emissivity are less than unity. Emissivity of a gray body is expressed as

$$\varepsilon= \,\frac{{Energy\,emitted\,by\,\,a\,gray\,body\,at\,temperature\,'T'}}{{Energy\,emitted\,by\,\,a\,black\,body\,at\,temperature\,'T'}} = \frac{E}{{{E_b}}}$$

$$\varepsilon \, = \,\frac{{\sigma {{\text{T}}^{\text{4}}}}}{{{\sigma _{\text{b}}}{{\text{T}}^{\text{4}}}}} = \frac{\sigma }{{{\sigma _{\text{b}}}}}$$

Value of Stefan-Boltzman's 'σ' constant depends upon nature of body, state of its surface and temperature. For a black body value of Stefan-Boltzman's 'σ' is equal to $5.67 \times 10^{-8}$ W/ $m^2$-$K^4$.

**Wien's Displacement Law:**

Wein's displacement law states that product of absolute temperature and wavelength at which emissive power is maximum, is constant. It has been established that monochromatic emissive power of a black body depends upon its temperature and wavelength of emitted radiations. For a given temperature, emissive power initially increases with increase in wavelength, attains a maximum value corresponding to particular wavelength and then decreases with further increases in wavelength of emitted radiations as discussed in the previous Lesson. With increase in temperature, maximum emissive power occurs at smaller wavelengths. Wein's displacement law gives the value of the wavelength at which emissive power of a body is maximum for a given temperature and is expressed as

$${(\lambda )_m}\,T = \,0.0029\,\,\,\,m - K$$

${(\lambda )_m}$ - Wavelength at which monochromatic power is maximum corresponding to temperature 'T'.

**Projected Area**

The area, $dA_1$, as seen from the prospective of a viewer, situated at an angle θ from the normal to the surface, will appear somewhat smaller, as $\cos\theta \cdot dA_1$. This smaller area is termed the projected area and has been shown in Figure 1.

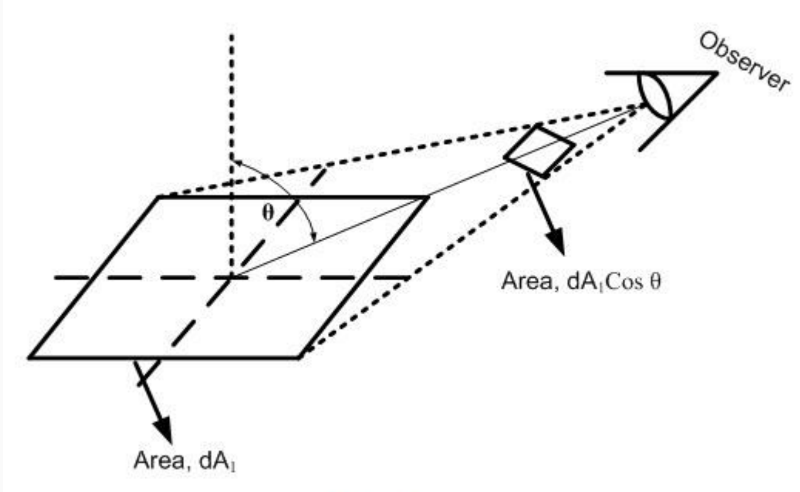



**Figure 1**

## Lesson-21 Solid Angle and Intensity of Radiation, Radiation Heat Transfer between Two Black Bodies

**Unit Solid Angle and Intensity of radiation:**

Generally speaking solid angle is that fraction of surface of a sphere that is seen by an observer positioned at the centre of a sphere. The ratio of area of this small surface being observed from centre of the sphere to square of radius of sphere represents solid angle. An observer standing at the centre of a sphere of radius 'R' will see a curve (which is fraction of surface of sphere)

The unit **solid angle** is defined as the angle covered by unit area on a surface of a sphere of unit radius when joined with the centre of the sphere and has been shown in Figure 1. Unit solid angle is measured in the steradians and is expressed as

$$\omega = \frac{A}{R^2}$$

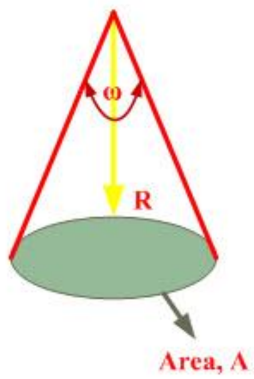



Figure 1

The **intensity of radiation** is defined as the rate of emission of radiation in a given direction from a surface per unit solid angle and per unit projected area of a radiating surface on a plane perpendicular to the direction of radiation.

$$E_b = \pi I_b$$

$E_b$ is the energy emitted and $I_b$ is the intensity of radiations.

**Radiation Heat Transfer between Two Black Bodies: Configuration Factor**

Radiation heat exchange between two bodies is influenced by the following parameters

**i) Temperature of the individual bodies**

**ii) Radiation properties of the individual bodies such as emissivity**

**iii) Orientation of the bodies relative to each other**

Orientation of bodies relative to each other means how well one body is able to see the other body. The effect of orientation of the bodies on radiation heat transfer is accounted by considering a factor called configuration factor which is also called shape or view factor. In order to understand the significance of configuration factor let us consider a body 1 which emitting radiations from its bottom surface only and a part of these radiations is intercepted by each of three bodies 2,3 and 4 as shown in Figure 2.

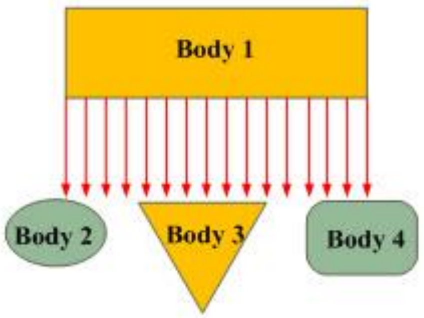



**Figure 2**

Configuration factor between body 1 and body 2 is denoted by $F_{1-2}$ and is expressed as

$$F_{1-2} = \frac{Radiation\ Energy\ received\ by\ body\ 2\ from\ body\ 1}{Total\ energy\ radiated\ by\ body\ 1}$$

Subscript 1 represents the emitting body and 2 represents the receiving or intercepting body.

Similarly, configuration or view or shape factor between body 2 and body 1 can be expressed as

$$F_{2-1} = \frac{Radiation\ Energy\ received\ by\ body\ 1\ from\ body\ 2}{Total\ energy\ radiated\ by\ body\ 2}$$

Similarly, configuration or view or shape factor between body 1 and body 3 can be expressed as

$$F_{1-3} = \frac{Radiation\ Energy\ received\ by\ body\ 3\ from\ body\ 1}{Total\ energy\ radiated\ by\ body\ 1}$$

Similarly, configuration or view or shape factor between body 1 and body 4 can be expressed as

$$F_{1-4} = \frac{Radiation\ Energy\ received\ by\ body\ 4\ from\ body\ 1}{Total\ energy\ radiated\ by\ body\ 1}$$

Configuration or view or shape factor depends upon the geometry / shape of the bodies involved in the radiation heat exchange and is independent of surface properties and temperatures of the bodies.

A mathematical expression for configuration or view or shape factor can be obtained by considering two black bodies 1 and 2 exchanging heat by radiations when maintained at temperatures $T_1$ and $T_2$ respectively. These two bodies are at a distance 'S' from each other and having areas $A_1$ and $A_2$ respectively as shown in Figure 3.

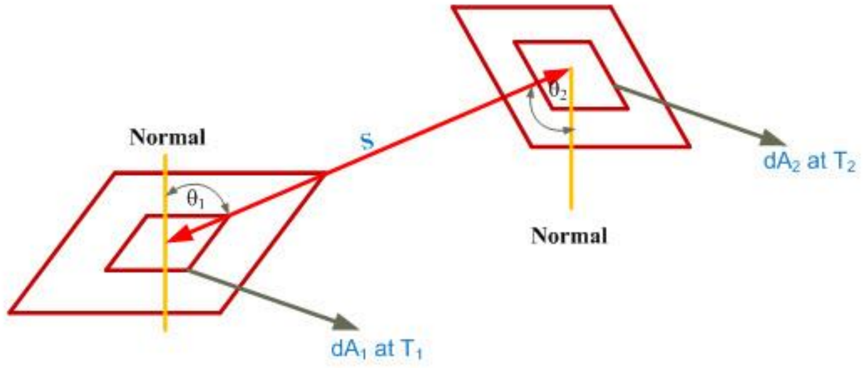



Figure 3

Let us consider two small elements of areas $dA_1$ and $dA_2$ at the centers of black bodies 1 and 2 respectively which are exchanging heat with each other by radiation.Heat radiated from element of area $dA_1$ towards element of area $dA_2$ is expressed as $dQ_{(1-2)}$ = Intensity of radiation of element of area $dA_1$ X projected area of element of area $dA_1$ along OP X solid angle made by element of area $dA_2$

$$= I_{b1}\,(dA_1 \cos\theta_1)\left(\frac{dA_2 \cos\theta_2}{S^2}\right) = I_{b1}\,(dA_1)(dA_2)\left(\frac{\cos\theta_1 \cos\theta_2}{S^2}\right)$$

This is the total heat lost by the element of area $dA_1$ and received by the element of area $dA_2$. Similarly heat is radiated by the element of area $dA_2$ which is at lower temperature ($T_2 < T_1$) and received by the element of area $dA_1$ is expressed as

$$dQ_{(2-1)} = I_{b2}\,(dA_1)(dA_2)\left(\frac{\cos\theta_1 \cos\theta_2}{S^2}\right)$$

:. Net heat radiated by the element of area $dA_1$ towards the element of area $dA_2$

$$dQ = dQ_{(1-2)} - dQ_{(2-1)}$$

$$= (dA_1)(dA_2)\left(\frac{\cos\theta_1 \cos\theta_2}{S^2}\right)[I_{b1} - I_{b2}] \qquad \text{..........(1)}$$

Since intensity of radiation is given by

$I_b = E_b/\pi = (\sigma T^4)/\pi$

Therefore, $I_{b1} = E_{b1}/\pi = (\sigma T_1^4)/\pi$ and $I_{b2} = E_{b2}/\pi = (\sigma T_2^4)/\pi$

Substituting the values of $I_{b1}$ and $I_{b2}$ in equation (1)

$$dQ = \frac{\sigma}{\pi}(T_1^4 - T_2^4)(dA_1)(dA_2)\left(\frac{\cos\theta_1 \cos\theta_2}{S^2}\right) \qquad (2)$$

In order to determine the total energy radiated from body 1 towards body 2, equation (2) is integrated over areas $A_1$ and $A_2$.

$$\therefore\ Q_{(1-2)} = \frac{\sigma}{\pi}(T_1^4 - T_2^4)\int_{A_1}\int_{A_2}\left(\frac{\cos\theta_1 \cos\theta_2 (dA_1)(dA_2)}{S^2}\right) \qquad (3)$$

where is total heat radiated from body 1 towards body 2

or Net = $F_{(1-2)} A_1\sigma (T_1^4 - T_2^4)$ (4)

Comparing equations (3) and (4), we get

$$\therefore\ F_{(1-2)} = \int_{A_1}\int_{A_2}\left(\frac{\cos\theta_1 \cos\theta_2}{A_1\, \pi S^2}\right)(dA_1)(dA_2) \qquad (5)$$

where $F_{(1-2)}$ is called the configuration factor or shape factor or view factor between the two radiating bodies and is a function of geometry only.

The subscripts 1 and 2 signify that the configuration factor is from body 1 to body 2. The configuration factor from body 2 to body 1 is given by

$$\therefore\ F_{(2-1)} = \int_{A_1}\int_{A_2}\left(\frac{\cos\theta_1 \cos\theta_2}{A_2\, \pi S^2}\right)(dA_1)(dA_2) \qquad (6)$$

The net energy lost by body 1 must be equal to net energy gained by body 2.

$$\therefore\ Q_{(1-2)} = -Q_{(2-1)} \qquad (\text{As } T_2 < T_1,\ \text{so -ve sign})$$

$$\therefore\ F_{(1-2)} A_1\sigma (T_1^4 - T_2^4) = -F_{(2-1)} A_2\sigma (T_2^4 - T_1^4)$$

$$\therefore\ F_{(1-2)} A_1 = F_{(2-1)} A_2 \qquad (7)$$

Equation (9) is known as reciprocal relation between the shape factors.

If Area $A_1$ is small compared with $A_2$, then

$$F_{(1-2)} = \int \frac{\cos\theta_1 \cos\theta_2}{\pi S^2}(dA2) \qquad (8)$$

**Lesson 22 Shape factor and its properties, Shape Factor of a Cavity with itself, Shape Factors of complex configurations derivable from perpendicular rectangles with common edge**

**Properties of Configuration Factor:**

Properties of shape factor are as:

1. The shape factor is purely a function of geometrical parameters.

2. When two bodies radiating energy with each other only, the shape factor relation is expressed as $A_1 F_{1-2} = A_2 F_{2-1}$ (1)

3. The shape factor of convex surface or flat surface with the other surface enclosing the first is always unity. This is because all the radiation coming out from the convex surface is intercepted by the enclosing surface but not vice versa.

4. A concave surface has a shape factor with itself because the radiation energy coming out from one part of the surface is intercepted by another part of the same surface. The shape factor of a surface with respect to itself is denoted by $F_{1-1}$.

$F_{1-1} = 0$ for convex and flat surface.

If a surface of area $A_l$ is completely enclosed by a second surface of area $A_2$ and if $A_l$ does not see itself ($F_{1-1} = 0$) then $F_{1-2} = l$

Then substituting the value of $F_{1-2} = l$ in equation (1) ,

$$F_{2-1} = \frac{A_1}{A_2}$$

Radiant exchange between coaxial cylinders, bodies placed in enclosures (sphere is kept in a room or box) are examples of this situation.

5. If n surfaces are taking part in radiation heat transfer then the energy radiated by one is always intercepted by the remaining (n-l) surfaces and by the surface itself also

$$F_{1-1} + F_{1-2} + F_{1-3} \ldots\ldots\ldots\ldots\ldots F_{1-n} = 1$$

$$F_{2-1} + F_{2-2} + F_{2-3} \ldots\ldots\ldots\ldots\ldots F_{2-n} = 1 \qquad (2)$$

: : :

: : :

$$F_{n-1} + F_{n-2} + F_{n-3} \ldots\ldots\ldots\ldots\ldots F_{n-n} = 1$$

In addition to the above equations, the reciprocal relation between any two surfaces also holds good

$F_{(1-2)} A_1 = F_{(2-1)} A_2$ or $F_{(1-3)} A_1 = F_{(3-1)} A_3$ and so on.

6. The shape factor between the surfaces $A_1$ and $A_2$ is equal to the sum of the shape factors between the surface $A_2$ and the surfaces which make the surface $A_1$, This point is illustrated as shown in Figure 1.

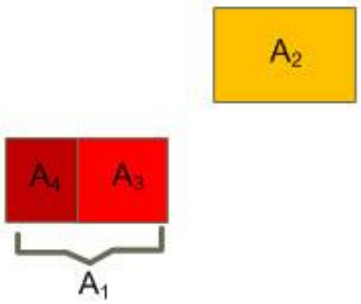



**Figure 1**

This states that the amount of radiated energy by $A_2$ and intercepted by $A_1$ is equal to the sum of the radiated energy intercepted by the areas $A_3$ and $A_4$ as shown in Figure.

Consider two surfaces of areas $A_1$ and $A_2$ are radiating heat to each other as shown in Figure1. Let $A_1$ be subdivided into $A_3$ and $A_4$ (i.e. $A_1 = A_3 + A_4$).

Then the radiant heat exchange between $A_1$ and $A_2$ is expressible as

$$Q_{1-2} = Q_{3-2} + Q_{4-2}$$

Considering the surfaces black

$$F_{(1-2)} A_1 \sigma (T_1^4 - T_2^4) = F_{(3-2)} A_3 \sigma (T_3^4 - T_2^4) + F_{(4-2)} A_4 \sigma (T_4^4 - T_2^4)$$

As $T_3 = T_4 = T_1$ and $A_3$ and $A_4$ are merely sub-divisions of $A_1$

Therefore $F_{(1-2)} A_1 = F_{(3-2)} A_3 + F_{(4-2)} A_4$ (3)

The above expression shows that

$$F_{1-2} \quad (F_{3-2} + F_{4-2})$$

For radiant exchange from $A_2$ to $A_1$ (divided into $A_3$ and $A_4$) one has

$$F_{(2-1)} A_2 = F_{(2-3)} A_2 + F_{(2-4)} A_2$$

Therefore $F_{(2-1)} = F_{(2-3)} + F_{(2-4)}$ (4)

7. If the interior surface of a completely enclosed space such as room is subdivided into n parts, each part having a finite area $A_1, A_2, A_3, A_4, \ldots\ldots\ldots\ldots A_n$, then

$$F_{1-1} + F_{1-2} + F_{1-3} \ \ldots\ldots\ldots\ldots\ldots F_{1-n} = 1$$

$$F_{2-1} + F_{2-2} + F_{2-3} \ \ldots\ldots\ldots\ldots\ldots F_{2-n} = 1$$

:    :    :

:    :    :

$F_{n-1} + F_{n-2} + F_{n-3}$ ..............$F_{n-n} = 1$

$$\sum_{j=1}^{j=n} F_{i-j} = 1 \quad \text{where } i = 1,2,3......, n. \tag{5}$$

The above representation admits the shape factors $F_{1-1}, F_{2-2}, F_{3-3}$,............... $F_{n-n}$, since some of the surface may see themselves if they are concave.

**Shape Factor of a Cavity with itself:**

Figure 2 shows an irregular cavity having an inner area $A_1$ and is covered by a flat surface of area $A_2$. Configuration factor equations for this arrangement is written as

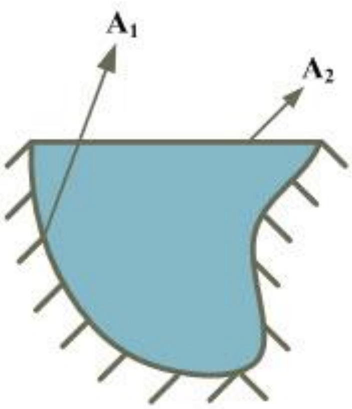



**Figure 2**

$$F_{1-1}+F_{1-2}= 1 \tag{6}$$

$$F_{2-2} + F_{2-1}=1 \tag{7}$$

Since the cavity is covered by a flat surface, so $F_{2-2} = 0$,

Substituting the value of $F_{2-2} = 0$ in equation (7), we get

$$F_{2-1} = 1 \tag{8}$$

The reciprocal relation between two surfaces is expressed as

$$F_{1-2} A_1 = F_{(2-1)} A_2$$

Using equation (8), the reciprocal relation can be written as

$$F_{1-2} = A_2 / A_1 \tag{9}$$

Substituting the values of $F_{1-2}$ in equation (1),

$$F_{1-1} = 1 - F_{1-2}$$

$$F_{1-1} = 1 - \frac{A_2}{A_1} \qquad (10)$$

Equation (10) represents configuration factor of a cavity with itself.

Therefore $F_{1-1} = 1 - F_{1-2} = 1 - \frac{A_2}{A_1} F_{2-1}$

Therefore $F_{1-1} = 1 - \frac{A_2}{A_1}(1 - F_{2-2}) = 1 - \frac{A_2}{A_1}$ as $F_{2-2} = 0$ (11)

The above expression is valid for all types of the cavities as shown in Figures 3 (a), (b), and (c).

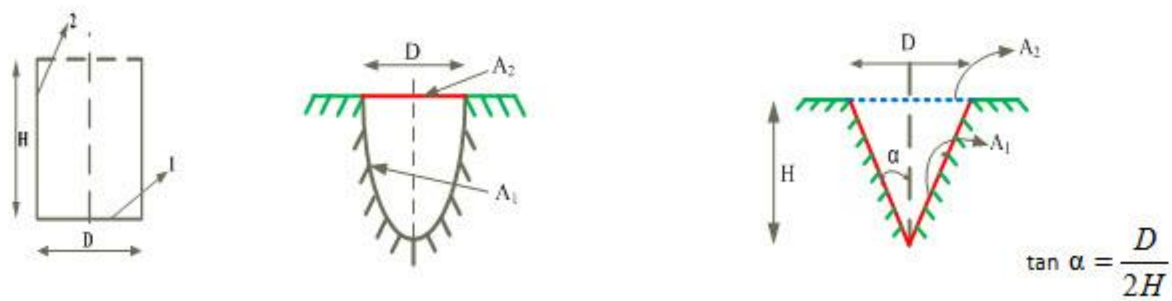



Figure 3 (a) Figure 3 (b) Figure 3 (c)

**(a) Shape factor of a cylindrical cavity of diameter D and height H with itself**

$$F_{1-1} = 1 - \frac{A_2}{A_1} \qquad (12)$$

From Figure 3 (a), it can be written that

$$A_2 = \frac{\pi D^4}{4}$$

$$A_1 = \pi DH + \frac{\pi D^4}{4}$$

Substituting the values of $A_1$ and $A_2$ in equation (12)

$$F_{1-1} = 1 - \frac{\frac{\pi D^4}{4}}{\pi DH + \frac{\pi D^4}{4}} \qquad (13)$$

**(b) Shape factor of a hemi-spherical cavity of diameter D with itself**

From Figure (b), it can be written that

$$A_2 = \frac{\pi D^4}{4}$$
$$A_1 = \frac{\pi D^4}{2}$$

Substituting the values of $A_1$ and $A_2$ in equation (12)

$$F_{1-1} = 1 - \frac{\frac{\pi D^4}{4}}{\frac{\pi D^4}{2}} = 1 - \frac{1}{2} = 0.5 \qquad (14)$$

**(c) Shape factor of a conical cavity of diameter D and height H with itself**

From Figure 3(c), it can be written that

$$A_2 = \frac{\pi D^4}{4} \quad , A_1 = \frac{\pi DL}{2} \quad \textit{where L is the slant height of the conical cavity}$$

Substituting the values of $A_1$ and $A_2$ in equation (12)

$$F_{1-1} = 1 - \frac{\frac{\pi D^2}{4}}{\frac{\pi DL}{2}} = 1 - \frac{D}{2L}$$
$$= 1 - 2\,Sin\alpha \qquad \textit{where } \alpha \textit{ is half vertex angle}$$
$$= 1 - \frac{D}{2\sqrt{H^2 + \frac{D^2}{4}}}$$
$$= 1 - \frac{D}{\sqrt{4H^2 + D^2}} \qquad (15)$$

**Shape Factors for Two Prependicular Plates:**

Determination of configuration factor for commonly used geometries such as parallel and perpendicular walls, parallel disks is cumbersome and complicated, therefore in such cases configuration factor is determined with the help of graphs.

**Shape factors for a system of Two Prependicular Plates**

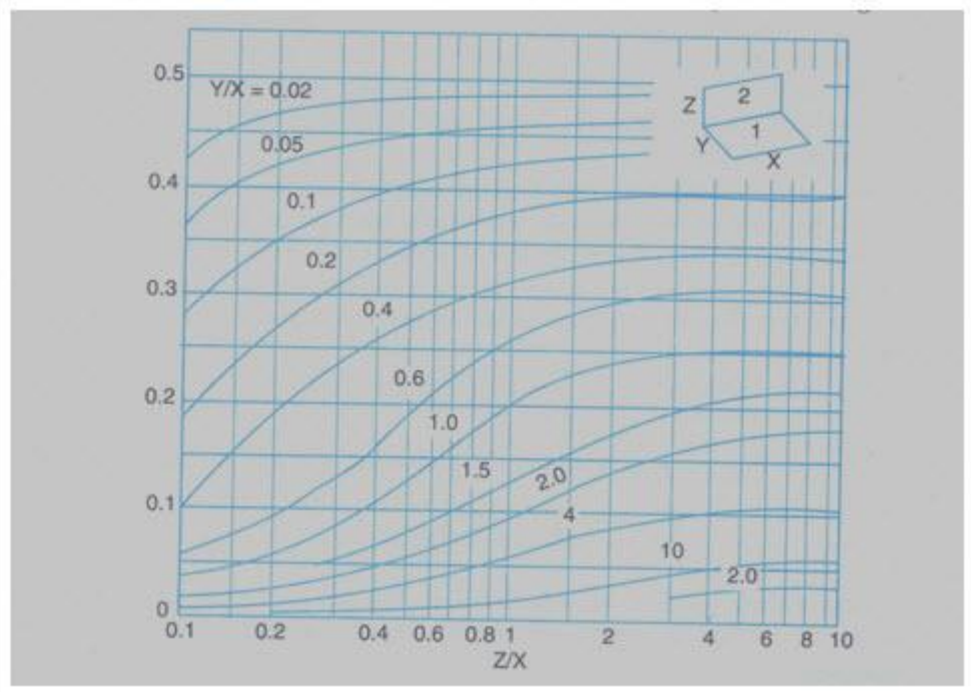



Figure 2 Configuration Factor for two perpendicular plates

**Complex Configurations Derivable From Perpendicular Rectangles with Common Edge:**

**i)** One rectangle is displaced from the common intersection line

$A_1 F_{1-4} = A_1 F_{1-3} + A_1 F_{1-2}$

Therefore $F_{1-2} = F_{1-4} - F_{1-3}$ (16)

The shape factors $F_{1-3}$ and $F_{1-4}$ may be found easily from the graph as shown in Figure 2.

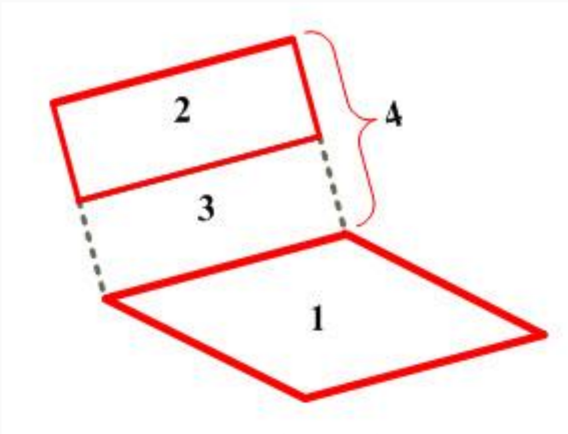



Figure 3

ii) Both rectangles are displaced from the common intersection line.

$A_1 F_{1-2} = A_5 F_{5-2} - A_3 F_{3-2}$

$= (A_5 F_{5-6} - A_5 F_{5-4}) - (A_3 F_{3-6} - A_3 F_{3-4})$

$= (A_5 F_{5-6} - A_5 F_{5-4} - A_3 F_{3-6} + A_3 F_{3-4})$

$= (A_5 F_{5-6} + A_3 F_{3-4}) - (A_5 F_{5-4} + A_3 F_{3-6})$ 17

The shape factors $F_{5\text{-}6}$, $F_{3\text{-}4}$, $F_{5\text{-}4}$ and $F_{3\text{-}6}$ may be found from graph as shown in Figure 2.

$$F_{1\text{-}2} = 1/A_1 \left[(A_5 F_{5\text{-}6} + A_3 F_{3\text{-}4}) - (A_5 F_{5\text{-}4} + A_3 F_{3\text{-}6})\right]$$

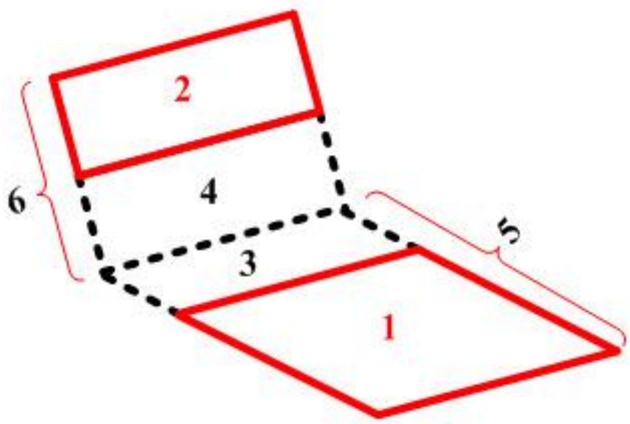



Figure 4

**iii)** The corner of both rectangles are touching at a point which lies on common intersection line.

$$A_1 F_{1\text{-}2} = A_6 F_{6\text{-}5} - A_1 F_{1\text{-}4} - A_3 F_{3\text{-}2} - A_3 F_{3\text{-}4} \quad (18)$$

The following reciprocal relation is valid for the above configuration.

$$A_1 F_{1\text{-}2} = A_3 F_{3\text{-}4}$$

Substituting this in the above equation (18)

$$F_{1\text{-}2} = 1/A_1 \left[A_6 F_{6\text{-}5} - A_1 F_{1\text{-}4} - A_3 F_{3\text{-}2}\right] \quad (19)$$

The shape factors $F_{6\text{-}5}$, $F_{1\text{-}4}$, and $F_{3\text{-}2}$ may be found from graph as shown in Figure 2.

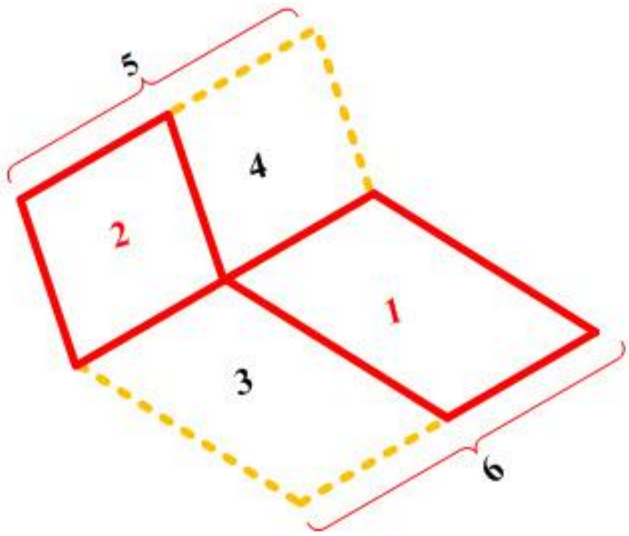



Figure 5

**Lesson-23 Radiation between two infinite parallel plates and proof of Kirchhoff's law of Radiation, Radiation Shield**

**Radiation between two infinite parallel plates and proof of Kirchhoff's law of Radiation:**

At a given temperature, the total emissive power of a any body is equal to its absorptivity multiplied by total emissive power of a perfect black body at that temperature.

Therefore $E = \alpha E_b$

But the ratio of total emissive power of any body to the total emissive power of a black body at the same temperature is called the emissivity of the body and is numerically equal to absorptivity.

$$\therefore \quad \alpha = \varepsilon = \frac{E}{E_b}$$

Consider two bodies C and D whose absorptivity are $\alpha_c$ and $\alpha_d$ as shown in Figure 1.

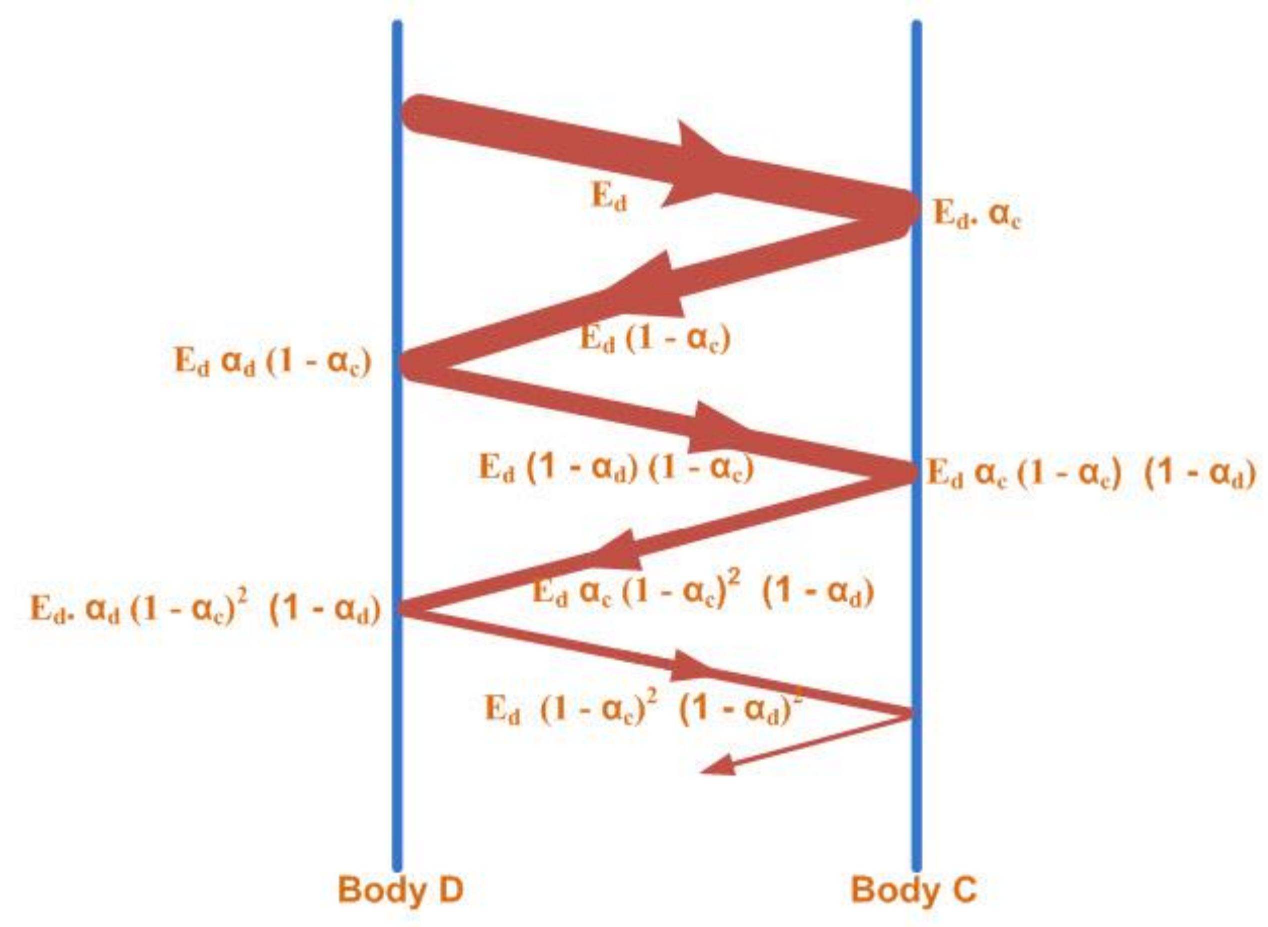



Figure 1

Considering the energy emitted by the body D.

(1) D emits the energy = $E_d$ (1)

(2) C absorbs energy = $E_d . \alpha_c$ (2)

and reflects energy=$E_d (1 - \alpha_c)$ (3)

(3) D absorbs energy = $E_d \alpha_d (1 - \alpha_c)$ (4)

and reflects energy = $E_d (1 - \alpha_d) (1 - \alpha_c)$ (5)

(4) C absorbs energy = $E_d \alpha_c (1 - \alpha_c) (1 - \alpha_d)$ (6)

and reflects energy = $E_d (1 - \alpha_c)^2 (1 - \alpha_d)$ (7)

(5) D absorbs energy = $E_d . \alpha_d (1 - \alpha_c)^2 (1 - \alpha_d)$ (8)

and reflects energy = $E_d (1 - \alpha_c)^2 (1 - \alpha_d)^2$ (9)

and so on upto times.

Considering the energy emitted by the body C.

(1) C emits the energy = $E_c$ (10)

(2) D absorbs energy = $E_c . \alpha_d$ (11)

and reflects energy=$E_c (1 - \alpha_d)$ (12)

(3) C absorbs energy = $E_c \alpha_c (1 - \alpha_d)$ (13)

and reflects energy = $E_c (1 - \alpha_c) (1 - \alpha_d)$ (14)

(4) D absorbs energy = $E_c \alpha_d (1 - \alpha_d) (1 - \alpha_c)$ (15)

and reflects energy = $E_c (1 - \alpha_d)^2 (1 - \alpha_c)$ (16)

(5) C absorbs energy = $E_c . \alpha_c (1 - \alpha_d)^2 (1 - \alpha_c)$ (17)

and reflects energy = $E_c (1 - \alpha_d)^2 (1 - \alpha_c)^2$ (18)

and so on upto infinite number of times.

Considering equations (1), (4), (8), (11), (I5), net energy lost by the body D

= Energy emittedby it – energy absorbedby it

$q_{(dc)} = E_d - [E_d \alpha_d (1 - \alpha_c) + E_d . \alpha_d (1 - \alpha_c)^2 (1 - \alpha_d) + \ldots] - [E_c . \alpha_d + E_c \alpha_d (1 - \alpha_d) (1 - \alpha_c) + \ldots]$

Assuming $(1 - \alpha_c) (1 - \alpha_d) = K$

$q_{(dc)} = E_d - E_d \alpha_d (1 - \alpha_c) [1 + K + K^2 + \ldots\ldots] - E_c . \alpha_d [1 + K + K^2 + \ldots]$

But $1 + K + K^2 + \ldots\ldots = (1 - K)^{-1}$

$q_{(dc)} = E_d - (1 - K)^{-1} [E_d \alpha_d (1 - \alpha_c) + E_c . \alpha_d]$

$$= E_d - \frac{[\alpha_d(1-\alpha_c)E_d + E_c\alpha_d]}{(1-K)} \qquad = \frac{E_d(1-K) - [\alpha_d(1-\alpha_c)E_d + E_c\alpha_d]}{(1-K)}$$

Substituting the values of K

$$q_{(dc)} = \frac{E_d[\alpha_d + \alpha_c - \alpha_d \alpha_c] - [\alpha_d(1-\alpha_c)E_d + E_c\alpha_d]}{[1-(1-\alpha_d)(1-\alpha_c)]} = \frac{E_d \alpha_c - E_c\alpha_d}{[\alpha_d + \alpha_c - \alpha_c\alpha_d]} \qquad (19)$$

**i) If originally both bodies are at same temperature**

Then $q_{(dc)} = 0$ from equation (18A)

$E_d a_c = E_c a_d$

Assuming C as black body

Then $\alpha_c = 1$

$$\alpha_d = \frac{E_d}{E_c}$$

as $E_c = E_b$ $\quad \therefore \alpha_d = \frac{E_d}{E_b}$

Subscription 'b' represents black body.

However, $E_d / E_b$ is the emissivity of body 'D' according to definition of emissivity.

$\varepsilon_d = \alpha_c$ This is the statement of Kirchoff's law and hence it is proved.

**ii) If both the bodies are at different temperatures**

Using equation (19)

$$q_{dc} = \frac{E_d \alpha_c - E_c\alpha_d}{[\alpha_d + \alpha_c - \alpha_c\alpha_d]}$$

**According to Stefan-Boltzman Law**

$E_d = \varepsilon_d \sigma_b T_1^4$

$E_c = \varepsilon_c \sigma_b T_2^4$

Substituting these values in equation (19), we get

$$q_{dc} = \frac{\varepsilon_d \varepsilon_c \sigma_b T_1^4 - \varepsilon_d \varepsilon_c \sigma_b T_2^4}{[\varepsilon_d + \varepsilon_c - \varepsilon_c \varepsilon_d]}$$

$$q_{dc} = \frac{\sigma_b \left(T_1^4 - T_2^4\right)}{\left[ 1/\varepsilon_c + 1/\varepsilon_d - 1 \right]} \qquad (20)$$

**Radiation Shields**

Generally, the shields are used for reducing the heat radiation from one plane to another plane. If the shield 3 is placed in between the two planes as shown in Figure 2 and considering they are at temperatures $T_1$, $T_2$ and $T_3$.

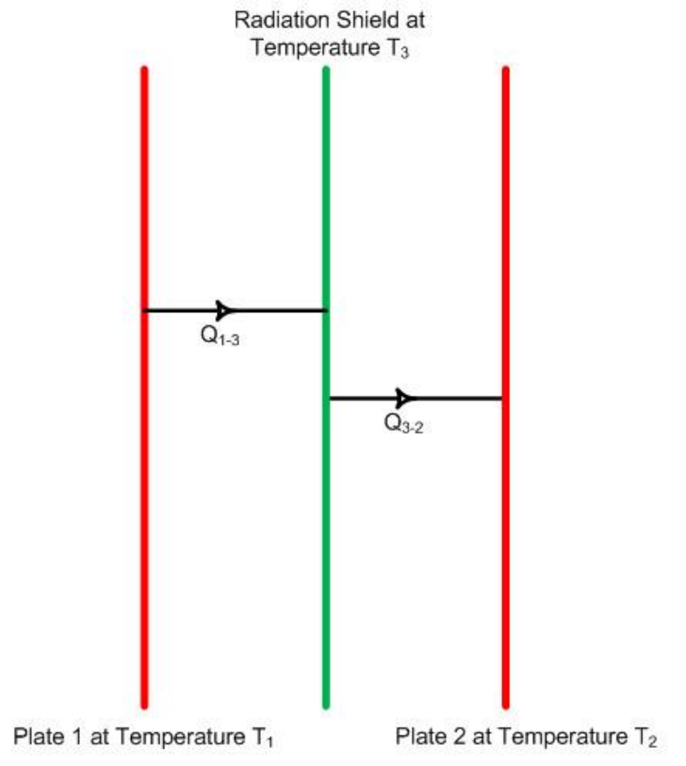



Figure 2

Assuming there is no temperature drop in the shield and considering the system is in steady state condition, we can write down the heat flow equation as

$$Q_{1\text{-}3} = \frac{A\sigma(T_1^4 - T_3^4)}{\left[\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3} - 1\right]} \qquad (21)$$

$F_{1\text{-}3}$= 1 as plates are parallel to each other

Similarly,

$$Q_{3\text{-}2} = \frac{A\sigma(T_3^4 - T_2^4)}{\left[\frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2} - 1\right]} \qquad (22)$$

For steady state conditions, $Q_{1\text{-}3}=Q_{3\text{-}2}$

$$\frac{(T_1^4 - T_3^4)}{\left[\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3} - 1\right]} = \frac{(T_3^4 - T_2^4)}{\left[\frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_2} - 1\right]} \qquad (23)$$

If $\varepsilon_1= \varepsilon_2= \varepsilon_3$, equation (23) becomes

$$T_3^4 = \frac{1}{2}\left(T_1^4 + T_2^4\right) \qquad (24)$$

Substituting the value of $T_3$ from equation (24) in equation (21), we get

$$Q_{1\text{-}3} = \frac{1}{2}\frac{A\sigma(T_1^4 - T_3^4)}{\left[\frac{2}{\varepsilon} - 1\right]} \qquad (25)$$

If there is no radiation shield present between plates 1 and 2, heat radiated is expressed as

$$Q_{1\text{-}2} = \frac{A\sigma(T_1^4 - T_2^4)}{\left[\frac{2}{\varepsilon} - 1\right]} \qquad (26)$$

Comparing equations (25) and (26), we get

$$Q_{1\text{-}2(\text{with shield})} = \tfrac{1}{2} \times Q_{1\text{-}2(\text{without shield})}$$

It means that with the addition of radiation shield, heat transfer rate is reduced to half of that of without the presence of radiation shield between two parallel bodies exchanging heat with each other by radiation. If ‘n’shields are present between the two radiating bodies, then the heat transfer will be expressed as

$$Q_{1-2(with\,'n'\,shields)} = \left(\frac{1}{n+1}\right)\frac{A\sigma(T_1^4 - T_3^4)}{\left[\frac{2}{\varepsilon} - 1\right]} \qquad (27)$$

## Lesson-24 Problems on Radiation

**Problem 7.7** **A furnace inside temperature of 2250 K has a glass circular viewing of 6 cm diameter. If the transmissivity of glass is 0.08, make calculations for the heat loss from the glass window due to radiation.**

**Solution:**

The radiation heat loss from the glass window is given by

$$Q = \sigma_b\, A\, T^4 \times \tau$$

Where $\tau$ is the transmissivity of glass

$$Q = 5.67 \times 10^{-8} \times \frac{\pi}{4}(0.06)^2 \times 2250^4 \times 0.08 = 328.53 \text{ W}$$

**Problem 7.4** **A thin metal plate of 4 cm diameter is suspended in atmospheric air whose temperature is 290 K. the plate attains a temperature of 295 K when one of its face receives radiant energy from a heat source at the rate of 2 W. If heat transfer coefficient on both surfaces of the plate is stated to be 87.5 W/m²-deg, workout the reflectivity of the plates.**

**Solution:**

Heat lost by convection from both sides of the plate

$$= 2h\, A\, \Delta t$$

The factors 2 accounts for two sides of the plate

$$= 2 \times 87.5 \times \left\{\frac{\pi}{4}(0.04)^2\right\} \times (295 - 290) = 1.1 \text{ W}$$

For most of solids, the transmissivity is zero.

$\therefore$ Energy lost by reflection $= 2.0 - 1.1 = 0.9$ W

Reflectivity $\rho = \frac{Q_r}{Q} = \frac{0.9}{2.0} = 0.45.$

**Problem 7.9** **A black body of total area 0.045 m² is completely enclosed in a space bounded by 5 cm thick walls. The walls have a surface area 0.5 m² and thermal conductivity 1.07 W/m-deg. If the inner surface of the enveloping wall is to be maintained at 215°C and the outer wall surface is at 30°C, calculate the temperature of the black body. Neglect the difference between inner and outer surfaces areas of enveloping material.**

**Solution:**

Net heat radiated by the black body to the enclosing wall,

$$Q_r = \sigma_b A(T_b^4 - T_w^4)$$

$$= 5.67 \times 10^{-8} \times 0.045(T_b^4 - 488^4)$$

Where $T_b$ is the temperature of the black body in degree kelvin.

Heat conducted through the wall,

$$Q_c = \frac{kA\,\Delta t}{\delta} = \frac{1.07 \times 0.5 \times (215 - 30)}{0.05} = 1979.5 \text{ W}$$

Under steady state conditions, the heat conducted through the wall must equal the net radiation loss from the black body. Thus

$$= 5.67 \times 10^{-8} \times 0.045(T_b^4 - 488^4) = 1979.5$$

$$T_b^4 = \frac{1979.5}{5.67\times10^{-8}\times0.045} + 488^4 = 8349.47 \times 10^8$$

∴ Temperature of the black body, $T_b$ =955.9 K

**Problem 7.12 A furnace radiation at 2000K. Treating it as a black body radiation, calculate the**

**(i) Monochromatic radiant flux density at 1 μm wavelength.**

**(ii) Wavelength at which emission is maximum and the corresponding radiant flux density**

**(iii) Total emissive power, and**

**(iv) Wavelength λ such that emission from 0 to λ is equal to the emission from λ to ∞.**

**Solution:**

(a) From Planck's law of distribution,

$$(E_\lambda)_b = \frac{C_1 \lambda^{-5}}{\exp\,[C_2/\lambda T]-1}$$

$$= \frac{0.374\times10^{-15}\times(1\times10^{-6})^{-5}}{\exp\,[1.4388\times10^{-2}/1\times10^{-6}\times2000]-1}$$

$$= \frac{0.374\times10^{15}}{1331.4-1} = \mathbf{2.81 \times 10^7 W/m^2}$$ per meter wavelength

(b) From Wien's displacement law; $\lambda_{max}\, T = 2.898 \times 10^{-3}$

$$\therefore \lambda_{max} = \frac{2.898\times10^{-3}}{2000} = \mathbf{1.449 \times 10^{-6} m}$$

Maximum radiant flux density,

$$1.285 \times 10^{-5} T^5 = 1.285 \times 10^{-5} \times (2000)^5$$

$$= \mathbf{4.11 \times 10^{11} W/m^2} \text{ per metre wavelength}$$

(c) From Stefan – Boltzman law,

$$E = \sigma_b T^4$$

$$= 5.67 \times 10^{-8} \times (2000)^4 = \mathbf{907200\ W/m^2}$$

(d) The emission in the band width 0 to λ is half of the total emission from 0 to ∞. Therefore, fractional emissive power is 0.5. Corresponding to $F_{0-\lambda} = 0.5$, the wavelength temperature product as read from table 7.1 is approximately 4100 μm-K. Thus λT = 4100 and so

$$\lambda = \frac{4100}{2000} = 2.05 \ \mu\text{m}$$

**Problem 7.16** **A polished metal pipe 5 cm outside diameter and 370 K temperature at the outer surface is exposed to ambient conditions at 295 K temperature. The emissivity of the surface is 0.2 and the convection coefficient of heat transfer is 11.35 W/m$^2$-deg. Calculate the heat transfer by radiation and natural convection per metre length of the pipe. Take thermal radiation constant**

**What would be the overall coefficient of heat transfer by the combined mode of convection and radiation?**

**Solution:**

Surface area A of the pipe per metre run is equal to

$$\pi dl = \pi \times 0.05 \times 1 = 0.157 \text{ m}^2$$

$$Q_r(\text{radiation}) = \in \sigma_b A\,(T_1^4 - T_2^4)$$

$$= 0.2 \times 5.67 \times 10^{-8} \times 0.157(370^4 - 295^4) = 19.88 \text{ W}$$

$$Q_c(\text{convection}) = h\,A\,\Delta t$$

$$= 11.35 \times 0.157(370 - 295)$$

$$= 133.64 \text{ W}$$

$$Q_t(\text{total}) = 19.88 + 133.64 = 153.52 \text{ W}$$

The total heat exchange can be expressed as:

$Q_t = U\,A\,\Delta T$ where U is the overall coefficient of heat transfer

$$\therefore U = \frac{Q_t}{A\,\Delta T} = \frac{153.52}{0.157(370-295)} = 13.04 \text{ W/m}^2 - \text{deg}$$

**Problem 7.18** **A gray surface has an emissivity at a temperature of 550 K source. If the surface is opaque, calculate its reflectivity for a black body radiation coming from a 550 K source.**

**(b)** **A small 25 mm square hole is made in the thin-walled door of a furnace whose inside walls are at 920 K. if the emissivity of the walls is 0.72, calculate the rate at which radiant energy escapes from the furnace through the hole to the room.**

**Solution:**

The requirement that all of the radiant energy striking any surface may be accounted for is:

$$\alpha + \rho + \tau = 1$$

Here:

(i) $\tau = 0$ as the surface is opaque

(ii) $\alpha = \in = 0.35$

This is in accordance with Kirchoff's law which states that absorptivity equals emissivity under the same temperature conditions.

$\therefore$ Reflectivity $\rho = 1 - (\alpha + \tau) = 1 - (0.35 + 0) = 0.65$

Thus the surface reflects 65 percent of incident energy coming from a source at 550 K.

(b) The small hole acts as a black body and accordingly the rate at which radiant energy leaves the hole is

$$E = \sigma_b A T^4$$
$$= 5.67 \times 10^{-8} \times (0.025 \times 0.025) \times 920^4$$
$$= \mathbf{25.38\ watts}$$

**Note:** The data about the emissivity of the inside wall is not needed.

**Problems on Radiation**

**Problem 1:** The temperature of the flame in a furnace is 1900 K, find (a) monochromatic energy emission at 1μ per m² (b) $\lambda_{max}$ (c) monochromatic energy emission at $\lambda_{max}$ and at 1900 K. (d) Ratio of $E_{b\lambda}$ / $(E_{b\lambda})_{max}$ (e) Total energy emitted/m².

**Solution:**

$$E_{b\lambda} = \frac{C_1 \lambda^{-5}}{\left[(e)^{C_2/\lambda T} - 1\right]}$$

As $\lambda = 1 = 10^{-6}$ m

$$E_{b\lambda} = \frac{3.7404 \times 10^{-16} (10^{-6})^{-5}}{\left[(e)^{\frac{0.014389}{10^{-6} \times 1900}} - 1\right]} = 1.91815 \text{ x } 10^{11}\ \text{W/m}^2 - \text{m}$$

(b) According to Wein's Law

$$\lambda_{max} = \frac{2900}{T} = \frac{2900}{1900} = 1.53\,\mu\ = 1.53 \text{ x } 10^{-6}\ \text{m}$$

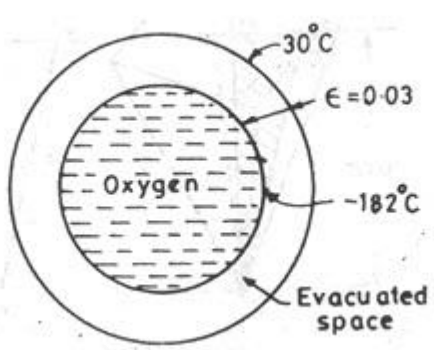
$$\text{( C) } (E_{b\lambda})_{max} = \frac{3.7404x10^{-16}(1.53x10^{-6})^{-5}}{\left[(e)^{\frac{0.014389}{1.53 \text{x} 10^{-6} \times 1900}} - 1\right]} = 3.183 \text{ x } 10^{11}\ \text{W/m}^2 \text{ - m}$$

$$\text{(d) Ratio of energy} = \frac{1.91815 \times 10^{11}}{3.183 \times 10^{11}} = 0.6048$$

(e) According to Stefan's Boltzmann's Law

$$E_b = \sigma T^4 = 5.678 \text{ x } \left(\frac{T}{100}\right)^4 = 5.678x\left(\frac{1900}{100}\right)^4 = 5.678x10^4(1.9) = 0.74 \text{ x } 10^6\ \text{W/m}^2$$

**Problem 2:** Liquid oxygen (boiling temperature = -182°C) is to be stored in a spherical container of 30 cm diameter. The system is insulated by an evacuated space between inner space and surrounding 45 cm *ID* concentric sphere. For both sphere ε = 0.03 and temperature of the outer sphere is 30°C. Estimate the rate of heat flow by radiation to the Oxygen in the container.




Determine the rate at which liquid oxygen would evaporate at -182 °C. Latent Heat of oxygen = 215 KJ/Kg.

**Solution:** The heat flow between the two concentric sphere by radiation is given by

$$Q_{1-2} = \frac{A_1 \sigma \left(T_1^4 - T_2^4\right)}{\frac{1-\varepsilon_1}{\varepsilon_1} + \frac{1}{F_{1-2}} + \left(\frac{1-\varepsilon_2}{\varepsilon_2}\right)\frac{A_1}{A_2}}$$

For concentric sphere $F_{12} = 1$ and $\frac{A_1}{A_2} = \left(\frac{d_1}{d_2}\right)^2$

$$\therefore \quad Q_{12} = \frac{\sigma(T_1^4 - T_2^4)A_1}{\frac{1}{\epsilon_1} + \left(\frac{d_1}{d_2}\right)^2 \left(\frac{1-\epsilon_2}{\epsilon_2}\right)} \qquad \text{...(a)}$$

where $T_1 = -182 + 273 = 91$ K and $T_2 = 30 + 273 = 303$ K

$$A_1 = 4\pi R_1^2 = 4\pi \left(\frac{15}{100}\right)^2 = 0.282 \text{ m}^2$$

Now substituting the values in equation (*a*)

$$Q_{12} = \frac{5.67\,[(0.91)^4 - (3.03)^4] \times 0.282}{\frac{1}{0.03} + \left(\frac{30}{45}\right)^2 \left(\frac{1-0.03}{0.03}\right)}$$

$$= \frac{5.67(0.686 - 82.289) \times 0.282}{33.3 + 14.52} = -\frac{5.67 \times 81.6 \times 0.282}{47.82}$$

$$= -2.72 \text{ W}$$

The –ve sign indicates that the heat flows from outside to inside.

Therefore rate of evaporation of oxygen = $\frac{2.72 \times 3600}{215 \times 1000}$ = 0.0455 Kg/hr.

**Problem 3:**Determine in W radiation heat loss from each meter of 20 cm diameter heating pipe when it is placed centrally in the brick duct of square section 30 cm side.

Temperature of pipe surface = 200°C

Brick duct temperature = 20°C

Emissivity of the pipe surface = 0.8

Brick duct emissivity = 0.9

Assume only radiation heat transfer between pipe and brick duct.

If the system is in steady state condition then find the surface heat transfer coefficient of the brick duct assuming the temperature of the surroundings of the duct is 10°C.

**Solution:** Using the formula

$$Q=\frac{\sigma A_1[T_1^4-T_2^4]}{\frac{1}{\epsilon_1}+\left(\frac{1}{\epsilon_2}-1\right)\frac{A_1}{A_2}}$$

$T_1=200+273=473°K$

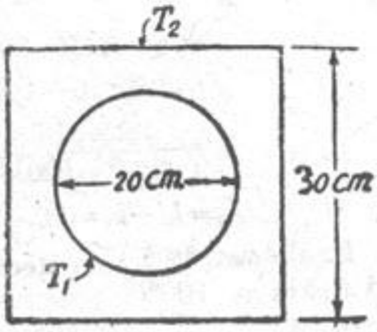
$T_2=20+273=293°K$

$\epsilon_1=0{\cdot}8$, and $\epsilon_2=0{\cdot}9$

$A_1=\pi D\times L$

(Surface Area of one metre pipe)

$=\pi\times0{\cdot}2\times1=0{\cdot}2\pi=0{\cdot}626\ m^2$




$A_2=$Perimeter$\times L$

(Surface Area of one metre duct)

$=4\times0{\cdot}3\times1=1{\cdot}2\ m^2$

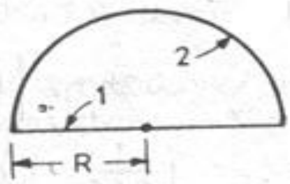
$$Q=\frac{5.678\times0.626[(4.73)^4-(2.93)^4]}{\frac{1}{0.8}+\left(\frac{1}{0.9}-1\right)\times\frac{0.626}{1.2}}=1187$$

If $T_3$ is the atmospheric temperature, then

$$Q=A_2 h(T_2-T_3)$$

$$h=\frac{Q}{A_2\ (T_2-T_3)}=\frac{1187}{1.2(20-10)}=98.9\ W/m^2-°C$$

**Problem 4:** For a hemispherical furnace, the flat floor is at 700°K and has an emissivity of 0.5. The hemispherical roof is at 1000 K and has emissivity of 0.25. Find the net radiative heat transfer from roof to floor.




**Solution:**

$$T_1 = 700 \text{ K and } \epsilon_1 = 0.5$$
$$T_2 = 1000 \text{ K and } \epsilon_2 = 0.25$$

$Q_{12}$ (from floor to roof).

$$= \frac{A_1\sigma\,(T_1^4 - T_2^4)}{\left(\frac{1}{\epsilon_1} - 1\right) + \frac{1}{F_{12}} + \left(\frac{1}{\epsilon_2} - 1\right)\frac{A_1}{A_2}}$$

In this case $A_1 = \pi R^2$ and $A_2 = \frac{4\pi R^2}{2}$

$$\therefore \quad \frac{A_1}{A_2} = \frac{\pi R^2}{2\pi R^2} = 0.5,\ F_{12} = 1$$

$$\therefore \quad Q_{12} = \frac{A_1\sigma\left[\left(\frac{700}{100}\right)^4 - \left(\frac{1000}{100}\right)^4\right]}{\left(\frac{1}{0.5} - 1\right) + 1 + \left(\frac{1}{0.25} - 1\right)\times 0.5}$$

$$= \frac{1 \times 5.67\ (7^4 - 10^4)}{20 + 1.5} \text{ if } A_1 = 1$$

$$= \frac{5.67\ (2401 - 10000)}{21.5} = -\ 2004 \text{ Watts}$$

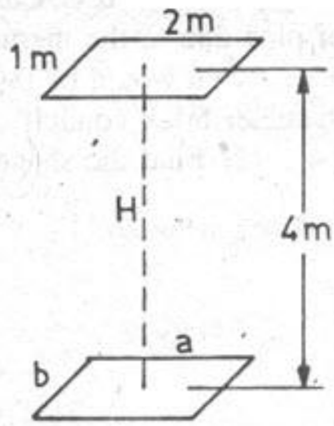
The – ve sign indicates that floor *gains the heat.*

$\therefore$ $Q_{12}$ = **2004 Watts/m²** (gained).

**Problem 5:** Two parallel rectangular surfaces 1 m × 2 m are opposite to each other at a distance of 4 m. The surfaces are black and at 100°C and 200°C respectively. Calculate the heat exchange by radiation between the two surfaces.

**Solution:**

The heat flow between the two surfaces is given by




$$Q_{12} = A_1 F_{12}\, \sigma\, (T_1^4 - T_2^4)$$

as $\quad \in_1 = \in_2 = 1$

$$B = \frac{a}{H} = \frac{2}{4} = 0.5$$

$$\text{and } C = \frac{b}{H} = \frac{1}{4} = 0.25$$

For the known values of $B$ and $C$, we can find $F_{12}$ from graph

$\therefore \quad F_{12} = 0.043$

$\therefore \quad Q_{12} = (2 \times 1) \times 0.043 \times 5.67\ [(4.73)^4 - (3.73)^4]$

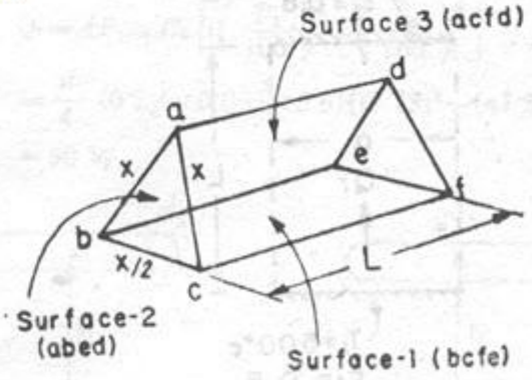
$= 2 \times 0.043 \times 5.67\ (500.5 - 193.5) =$ **149.7 watts.**

**Problem 6: Concider a triangular duct of length L as shown in Figure. For the given dimensions, prove that**

$F_{2-3}=0{\cdot}75$

$ab=ac=x$ and $bc=x/2$.

**Solution:**




From the diagram we can consider three surfaces as shown in figure.

$$F_{1-2}+F_{1-3}=1 \quad \ldots(a)$$
$$F_{2-1}+F_{2-3}=1 \quad \ldots(b)$$
$$F_{3-1}+F_{3-2}=1 \quad \ldots(c)$$

By symmetry $F_{1-2}=F_{1-3}$.

$\therefore\quad F_{1-2}=0{\cdot}5$ and $F_{1-3}=0{\cdot}5$ [from equation $(a)$]

$$\therefore\quad F_{2-3}=1-F_{2-1} \quad \ldots(d)$$

But $\quad F_{1-2}A_1=F_{2-1}A_2$

$$\therefore\quad F_{2-1}=F_{1-2}\cdot\frac{A_1}{A_2}=0{\cdot}5\times\frac{\left(\dfrac{x}{2}\times L\right)}{x\times L}=0{\cdot}25.$$

Substituting the value of $F_{2-1}$ in equation $(d)$

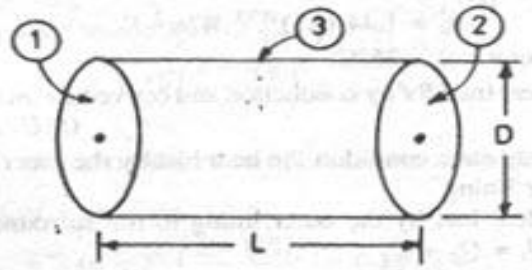
$$\therefore\quad F_{2-3}=1-0{\cdot}25=0{\cdot}75.$$

**Problem 7:** The radiation shape factor of the circular surfaces of a thin hollow cylinder of 10 cm diameter and 10 cm length is 0.1716. What is the shape factor of the curved surface of the cylinder with respect to itself.

**Solution:** The three surfaces are shown in Fig.

The given data is

$$F_{12} = F_{21} = 0.1716 \text{ as } A_1 = A_2$$




The shape factor relation among the three surfaces is given by

$$F_{11} + F_{12} + F_{13} = 1 \qquad ...(a)$$

and $$F_{33} + F_{32} + F_{31} = 1 \qquad ...(b)$$

But $$F_{31} = F_{32} \qquad ...(c)$$

But $$F_{11} = F_{22} = 0$$

Substituting (*c*) into (*b*), we get

$$F_{33} + F_{31} + F_{31} = 1$$

$$\therefore \quad F_{33} = (1 - 2F_{31}) \qquad ...(d)$$

Writing the relation between 1 and 3

$$A_1F_{13} = A_3F_{31}$$

$$\therefore \quad F_{31} = F_{13} \cdot \frac{A_1}{A_3} = F_{13} \cdot \frac{\frac{\pi}{4}D^2}{\pi DL} = F_{13}\frac{D}{4L} \qquad ...(e)$$

From equation (*a*)

$$F_{13} = 1 - F_{12} \text{ as } F_{11} = 0$$
$$= 1 - 0.1716 = 0.8284 \qquad ...(f)$$

as $F_{12} = 0.1716$ (given)

Now substituting from (*f*) into (*e*)

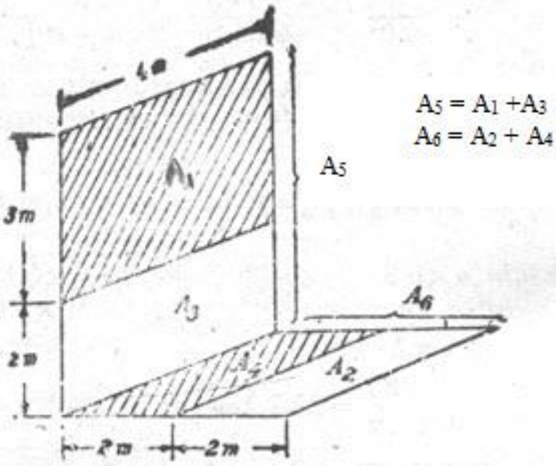
$$F_{31} = 0.8284 \times \frac{10}{4 \times 10} = 0.2071 \qquad ...(g)$$

Now substituting from (*g*) into (*d*)

$$F_{33} = 1 - 2 \times 0.2071 = 1 - 0.4142 = \mathbf{0.5858}$$

**Problem 8: Find the shape factor between $A_1$ and $A_2$ as shown in Figure and net heat transfer between $A_1$ and $A_2$.**

If $T_1$=400°C and $T_2$=200°C and both surfaces are black.




**Solution:**

$$F_{1-2}=F_{1-6}-F_{1-4}$$
$$=F_{6-1}\ \frac{A_6}{A_1}-F_{4-1}\cdot\frac{A_4}{A_1}$$
$$=\frac{A_6}{A_1}[F_{6-5}-F_{6-3}]-\frac{A_4}{A_1}[F_{4-5}-F_{4-3}]$$

From the graph we can find the values of $F_{6-5}$, $F_{6-3}$, $F_{4-5}$ and $F_{4-6}$ as

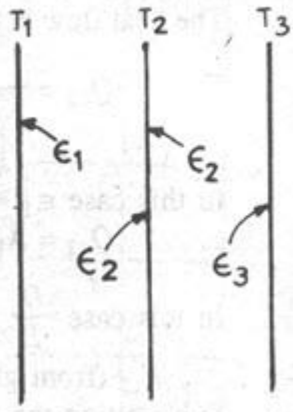
$$F_{6-5}=0{\cdot}2,\ F_{6-3}=0{\cdot}138,\ F_{4-5}=0\ 29,\ F_{4-3}=0{\cdot}225$$

$$\therefore \qquad F_{1-2}=\frac{4\times4}{4\times3}(0{\cdot}2-0{\cdot}138)-\frac{2\times4}{3\times4}(0{\cdot}290-0\ 225)$$
$$=0{\cdot}084-0{\cdot}040=0{\cdot}044.$$

(b) $$Q=A_1\epsilon\sigma(T_1^4-T_2^4).F_{1-2}$$
$$= (3 \times 4) \times 1.0 \times 5.678\ [(6.73)^4 - (4.73)^4\ ] \times 0.044$$
$$= 4664 \text{ W}$$

**Problem 9:** Calculate the net radiant heat exchange per $m^2$ area for two large parallel planes at temperatures of 427°C and 27°C respectively. $\epsilon$ (hot plane) = 0.9 and $\epsilon$ (cold plane) = 0.6.

If a polished aluminium shield is placed between them, find the precentage reduction in the heat transfer $\epsilon$ (shield) = 0.04.




**Solution:**

(a) In the absence of the shield the heat flow between the plates 1 and 3 is given by

$$Q_{13}=\frac{\sigma\ (T_1^4-T_3^4)}{\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}-1}$$

$$= \frac{5.67\left[\left(\frac{700}{100}\right)^4 - \left(\frac{300}{100}\right)^4\right]}{\frac{1}{0.9} + \frac{1}{0.6} - 1} = \frac{5.67\ (2401 - 81)}{1.11 + 1.67 - 1}$$

$$= \frac{5.67 \times 2320}{1.78} = \mathbf{7390\ Watts}$$

(b) When shield is placed between the plates 1 and 3, then

$$Q_{12} = Q_{23}$$

$$\therefore \quad \frac{\sigma\ (T_1^4 - T_2^4)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1} = \frac{\sigma\ (T_2^4 - T_3^4)}{\frac{1}{\epsilon_2} + \frac{1}{\epsilon_3} - 1}$$

$$\frac{\left(\frac{700}{100}\right)^4 - \left(\frac{T_2}{100}\right)^4}{\frac{1}{0.9} + \frac{1}{0.04} - 1} = \frac{\left(\frac{T_2}{100}\right)^4 - \left(\frac{300}{100}\right)^4}{\frac{1}{0.04} + \frac{1}{0.6} - 1}$$

$$\frac{2401 - x^4}{1.11 + 25 - 1} = \frac{x^4 - 81}{25 + 1.67 - 1} \text{ where } x = \left(\frac{T_2}{100}\right)$$

$$2401 - x^4 = \frac{25.11}{25.67}(x^4 - 81) = 0.9782\ x^4 - 79$$

$$\therefore \quad 1.9782\ x^4 = 2480 \quad \therefore \quad x^4 = 1253.8 \quad \therefore \quad x = 5.95$$

$$\frac{T_2}{100} = 5.95 \quad \therefore \quad \mathbf{T_2 = 595°K}$$

$$\therefore \quad Q_{12} = \frac{\sigma\ (T_1^4 - T_2^4)}{\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} - 1}$$

$$= \frac{5.67\left[\left(\frac{700}{100}\right)^4 - \left(\frac{595}{100}\right)^4\right]}{25.11}$$

$$= \frac{5.67\ (2401 - 1253)}{25.11} = \mathbf{2118\ Watts.}$$

$\therefore$ Reduction in heat flow due to shield

$= Q_{13} - Q_{12} = 7390 - 2118 = 5272$ watt.

$$\%\ \text{reduction} = \frac{5272}{7390} \times 100 = \mathbf{71.34\%}.$$

**Problem 10:** Determine the number of shields required to keep the temperature of the outside surface of a hollow brick lining of a furnace at 100°C when the temperature of the inside surface of the lining is 500°C. Take the emissivity of brick lining as well as for shield as 0.87

Heat transfer to the surrounding from the outer surface takes place by radiation and convection. The heat transfer coefficient for natural convection is given by

$h_a = 1.44\,(\Delta T)^{0.33}$ W/m²-K

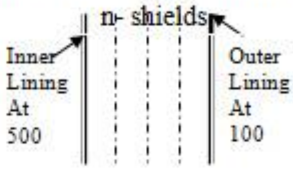
$T_a$ (air-temperature) = 25°C.

Neglect the heat transfer by conduction and convection between the brick lining.

**Solution:** For steady state condition the heat lost by the inner lining through shields to the outer lining

= Heat lost by the outer lining to the surrounding = Qt = Qc + Qr

where Qc and Qr are heats lost by outer lining to the surrounding by convection and radiation respectively.




$$Q_c = Ah_a\,(T_o - T_a)$$
$$h_a = 1.44(100-25)^{0.33} = 1.44 \times (75)^{0.33}$$
$$= 1.44 \times 4.16 = 5.99\ W/m^2\text{-}K$$
$$\therefore\quad Q_c = 1 \times 5.99(100-25) \text{ where } A = 1\ m^2$$
$$= 5.99 \times 75 = \mathbf{449.25\ W}$$

$$Q_r = A\sigma\epsilon\ (T_o{}^4 - T_a{}^4) = 1 \times 5.67 \times 0.87\left[\left(\frac{373}{100}\right)^4 - \left(\frac{298}{100}\right)^4\right]$$
$$= 5.67 \times 0.87(193.6 - 78.9)$$
$$= \mathbf{565.8\ W}$$

$$\therefore\quad Q_t = 449.25 + 565.8 = \mathbf{1055} \qquad ...(a)$$

This is the amount of heat radiated from the inner surface to outer surface through '$n$' shields.

When these are $n$ shields, the surface resistances are $(2n + 2)$ and space resistances are $(n + 1)$.

The space resistance for parallel plates

$$= \frac{1}{F_{12}} = \frac{1}{F_{23}} = \ldots\ldots = \frac{1}{F_{(n-1)-n}}$$

But $F_{12} = F_{23} = F_{(n-1)-n} = 1$

$$\therefore\quad \text{Total space resistance } (R_s) = (n+1)\,.\,\frac{1}{F_{12}} = (n+1)$$

Total surface resistances $(R_{sur})$

$$= 2\,(n+1)\left(\frac{1-\epsilon_1}{\epsilon_1}\right) \text{ as } \epsilon_1 = \epsilon_2 = \ldots\ldots = \epsilon_n \text{ (given)}$$

where $\dfrac{1-\epsilon_1}{\epsilon_1}$ is the radiation surface resistance between the two parallel surfaces, each of 1 $m^2$ area.

$$\therefore\quad R_{sur} = 2\,(n+1)\left(\frac{1-0.87}{0.87}\right) = 0.15\,(n+1)$$

$$Q_t = \frac{\sigma(T_1{}^4 - T_2{}^4)}{R_s + R_{sur}} = \frac{5.67\left[\left(\frac{773}{100}\right)^4 - \left(\frac{373}{100}\right)^4\right]}{(n+1) + 0.15\,(n+1)} \qquad ...(b)$$

Substituting the value of $Q_t$ from equation $(a)$ into $(b)$

$$1055 = \frac{5.67[(7.73)^4 - (3.73)^4]}{1.15(n+1)}$$

$$\therefore\quad (n+1) = \frac{5.67(3570.5 - 193.5)}{1.15 \times 1055} = 15.78$$

$$n = 15.78 - 1 = 14.78 \simeq \mathbf{15\ (shields)}$$

**Module 4. Heat Exchangers**

## Lesson-25 Introduction, Classification of Heat Exchangers, Logarithmic mean temperature difference

A heat exchanger is a device used for efficient transfer of heat between a hot and cold fluid when it is required to heat up a cold fluid or cool down a hot fluid. Both the fluids involved in heat exchange process are generally separated by a solid wall. Heat exchangers are used in wide range of applications such as

- Heating and air conditioning systems
- Automobile radiators
- Cooling of internal combustion engines by a coolant
- Boilers and condensers of a power plant
- Chemical plants
- Petroleum refineries

**Classification of Heat Exchangers:**

In order to meet the different and specific requirements of heat exchange between two fluids, different types of heat exchangers have been designed. However, heat exchangers are generally classified on the basis of following

- Nature of Heat Exchange Process
- Relative Direction of Flow of Fluids
- Mechanical Design of Heat Exchanging Surface
- Physical State of Heat Exchanging Fluids

**A) Nature of Heat Exchange Process:**

Depending upon the nature of heat exchange process, heat exchangers are categorized as

i) **Direct Contact Heat Exchanger:** In such a heat exchanger, hot and cold fluids exchange thermal energy by their physical mixing and there is a simultaneous transfer of heat and massas shown in Figure 1. Water cooling towers, Jet steam condensers are examples of direct contact heat exchanger.

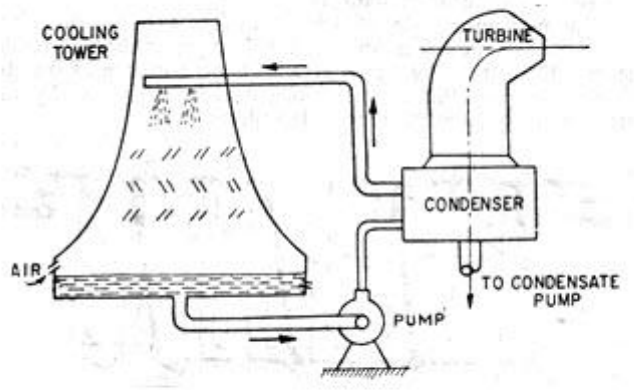



**Figure 1**

ii) **Regenerator Type of Heat Exchanger:** In regenerator heat exchanger, hot and cold fluids are brought in contact with a medium alternately with which they exchange heat. This is also known as storage type heat exchanger because heat transfer from hot fluid to cold fluid occurs through a coupling medium in the form of a solid matrix having a high heat capacity The hot fluid and cold fluid alternately flow through the matrix, the hot fluid storing heat in it and the cold fluid extracting heat from it. The arrangement is shown in Figure 2. During the first part of the cycle when hot fluid flows through the matrix, valves I and 4 are open and 2 and 3 are closed while during the latter part, valves 1 and 4 are closed and 2 and 3 are open.

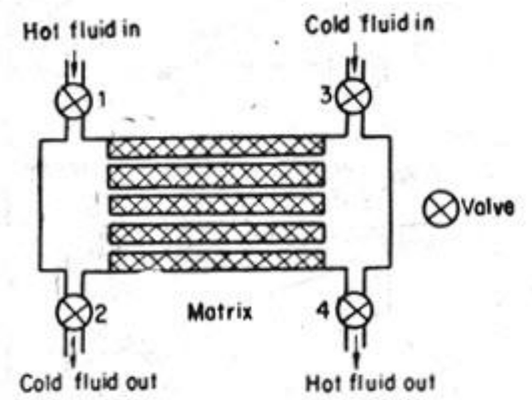



**Figure 2**

iii) **Recuperator:**-In this type of heat exchanger, hot and cold fluids do not mix and exchange heat through a separating wall which also offers thermal resistance. The heat transfer process in such heat exchangers consists of

- **Convective heat transfer from hot fluid to wall**
- **Conductive heat transfer from hot to cold surface of wall**
- **Convective heat transfer from wall to cold fluid**

Examples:- Boilers, automobiles, radiators

**B) Relative Direction of Flow of Fluids:**

Depending upon the relative direction of flow of fluids, heat exchangers are categorized as

i) **Parallel Flow:** Hot and cold fluids enter the heat exchanger from the same side, flow in same direction and leave in same direction as shown in Figure 3.


**Figure 3**

ii) **Counter Flow:** Hot and cold fluid enter from opposite side of heat exchanger, flow in opposite direction and leave in opposite direction as shown in Figure 4.


**Figure 4**

iii) **Cross Flow:** Hot and cold fluid flow in direction right angle to each other. If air is one of the fluids, pure counter flow is generally not preferred as shown in Figure 5.

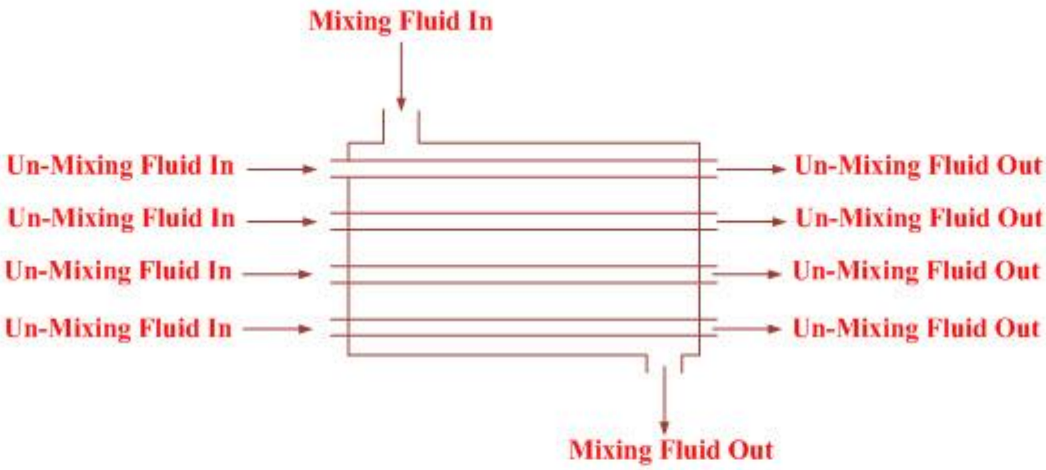



**Figure 5**

**C) Mechanical Design of Heat Exchange Surface**

i) **Concentric Tubes:** Two concentric tubes are used, each carrying one of the fluids.

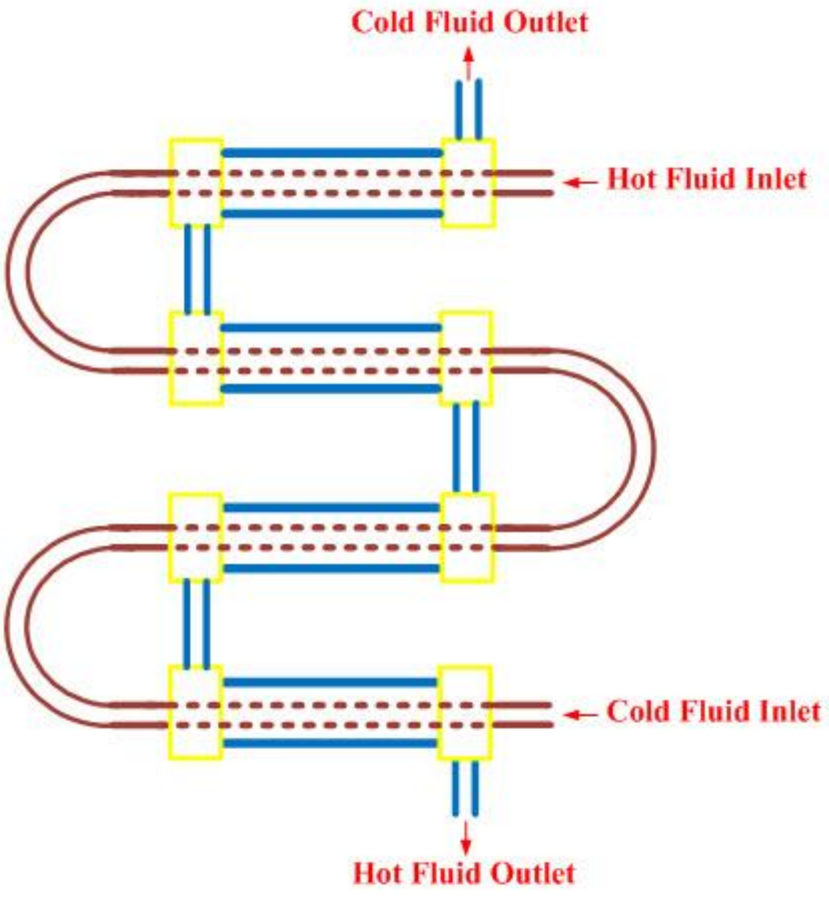



Figure 6

ii) **Shell & Tube:-** One of the fluid is carried through the bundle of tubes enclosed by a shell. The other fluid is forced through the shell and flows over the outside surface of the tubes as shown in Figure 7.

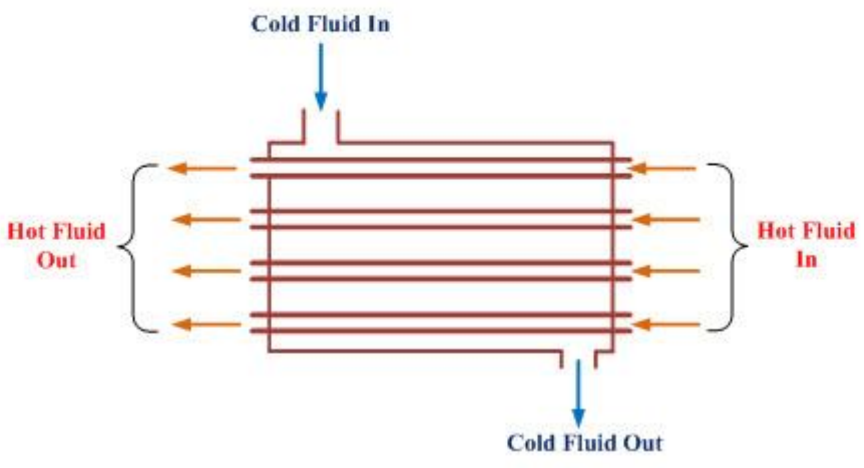



Figure 7

iii) **Multi shell and Tube Passes:-** Baffles are used to make the multi shell and Tube Passes heat exchanger as shown in Figure 8.

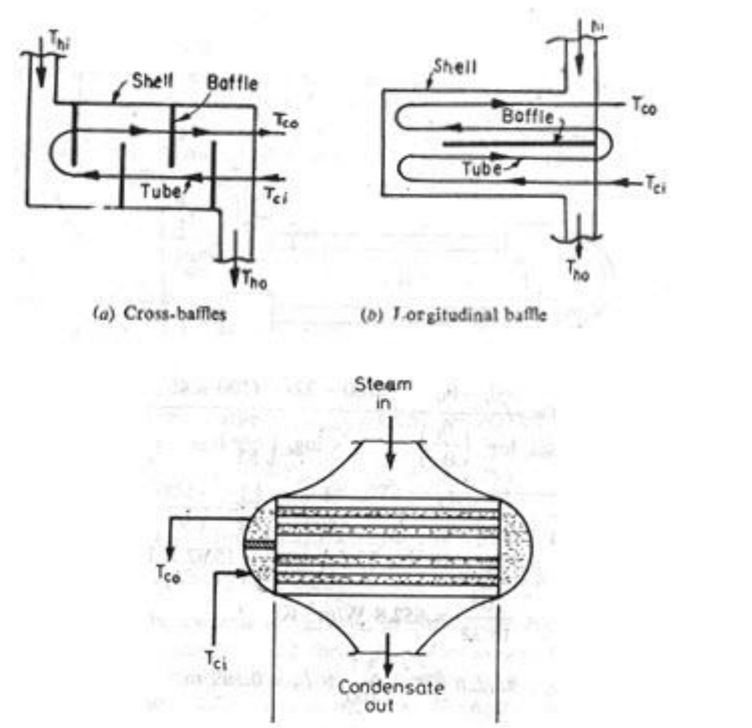



**Figure 8**

**D) Physical State of Heat Exchanging Fluids:**

Depending upon physical state of heat exchanging fluids, heat exchangers are categorized as

i) **Condenser**: Temperature of hot fluids remains constant all along the length of heat exchanger as it loses its latent heat while temperature of cold fluid increases

ii) **Evaporator:** Temperature of cold fluid remains constant as it gains its latent heat while temperature of hot fluid decreases.

**Logarithmic Mean Temperature Difference**

In a heat exchanger, thermal potential, responsible for heat exchange between the hot and cold fluids, changes as temperature of both the fluids changes along the length of the heat exchanger. In a parallel heat exchanger, thermal potential in a parallel heat exchanger is maximum at inlet; it decreases along the length and is minimum at the exit as shown in Figure 9. In a counter flow heat exchanger, maximum thermal potential exists at exit of the heat exchanger as shown in Figure 10.

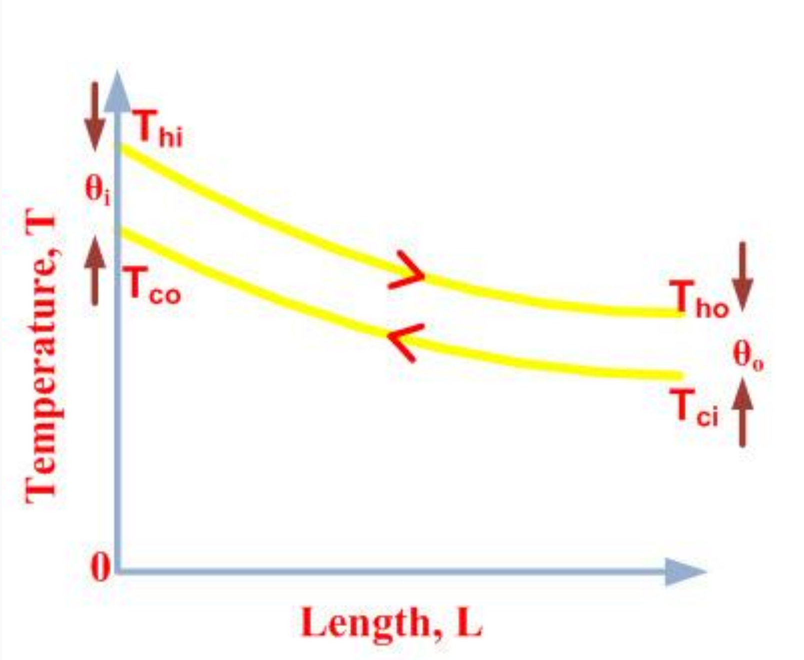



**Figure** 9 **Parallel Flow Heat Exchanger**

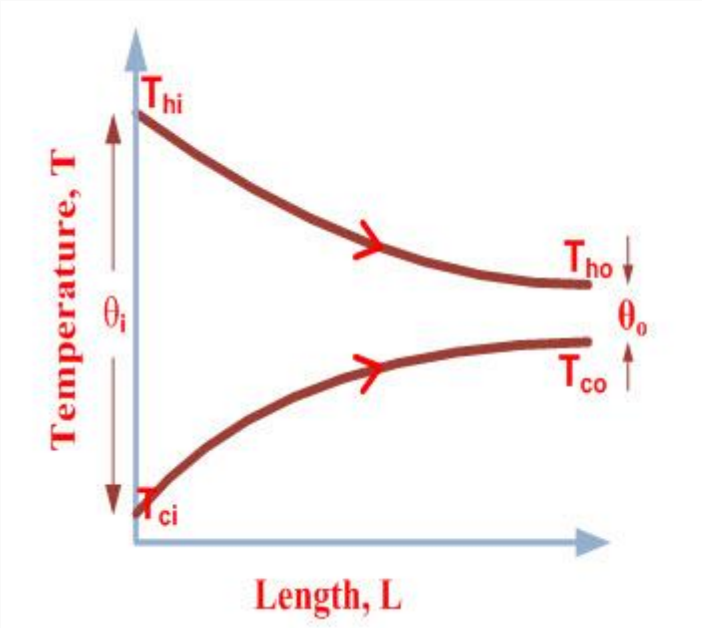



**Figure 10 Counter Flow Heat Exchanger**

However, in some heat exchangers such as condensers and evaporators, change in thermal potential along the length occurs only on account of change in temperature one of the fluids while temperature of the other fluid remains constant. Figures 11 and 12 represent variation of thermal potential along the length of a condenser and an evaporator

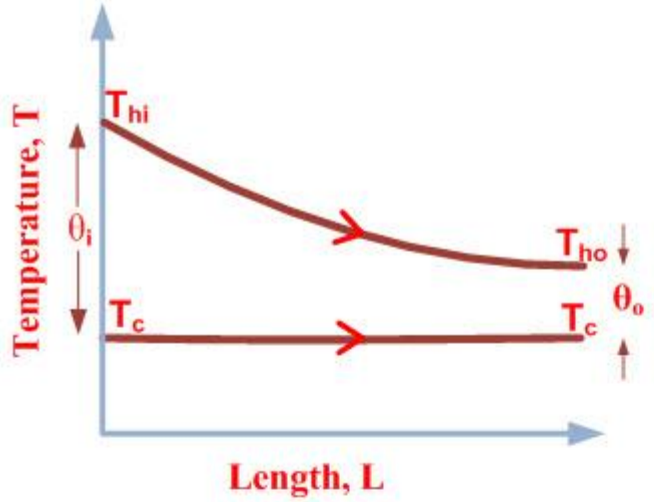



**Figure 11 Condenser**

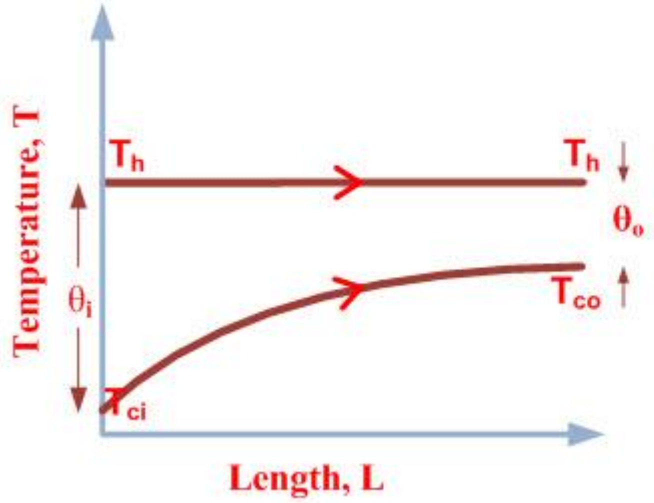



**Figure 12 Evaporator**

Heat transfer rate in a heat exchanger at a particular length is calculated by considering thermal potential at that length which is expressed as

$$Q = UA\,(T_h - T_c) \qquad (1)$$

However, overall heat transfer in a heat exchanger is calculated by using average thermal potential and is generally represented by term logarithmic temperature difference. Therefore, overall heat transfer in heat exchanger can be expressed as

$$Q = UA\,(\Delta T)_m \qquad (2)$$

Where

U is overall heat transfer coefficient, W/ ($m^2$-K)

A is heat transfer area, $m^2$

$(\Delta T)_m$ is mean temperature difference or logarithmic temperature difference, K

**Lesson-26 Logarithmic mean temperature difference for parallel flow heat exchanger, counter and cross flow heat exchangers, Overall Heat Transfer Coefficient for a Plane wall and Double Pipe Heat Exchanger**

**Overall Heat Transfer Coefficient**

1. **Plane Wall:**

Consider a wall of thickness 'δ' made of a material having thermal conductivity 'k' and temperatures of wall surfaces are $T_{s1}$ and $T_{s2}$. On one side of the wall, a hot fluid at temperature $T_h$ is flowing and on the other side a cold fluid at temperature $T_c$ is flowing as shown in Figure 1. Let $h_i$ and $h_o$ are convective heat transfer coefficients of hot and cold fluids with wall surfaces respectively. Under steady state conditions, heat transferred from hot fluid to wall surface is equal to heat conducted through the wall and it is equal to heat convected to the cold fluid from wall surface.

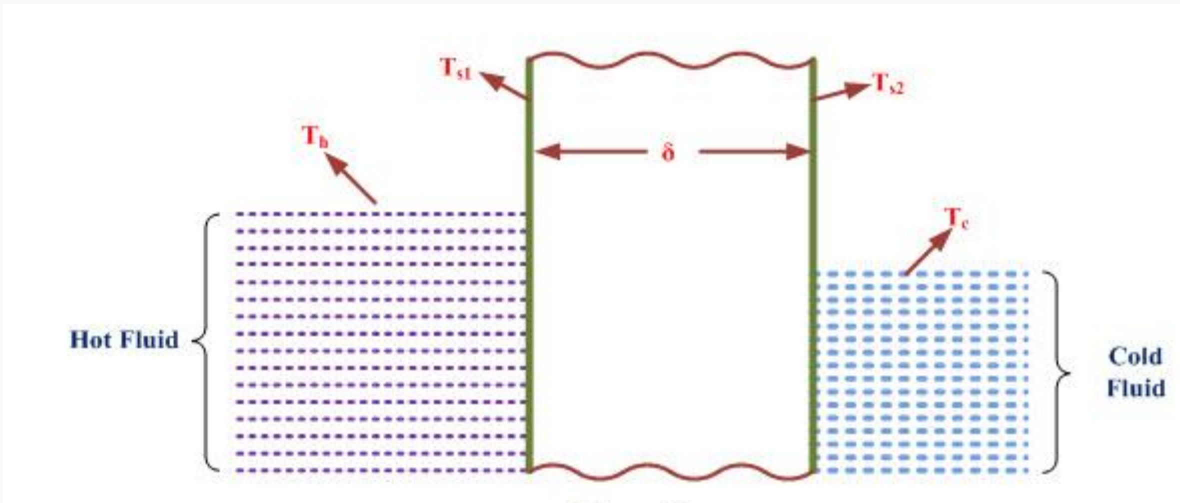



**Figure 1**

Heat transferred to wall surface by hot fluid is expressed as

$$Q = h_i A (T_h - T_{s1})$$

$$\frac{Q}{h_i A} = (T_h - T_{s1}) \qquad (1)$$

Heat transferred through the wall by conduction is expressed as

$$Q = kA\frac{(T_{s1} - T_{s2})}{\delta}$$

$$\frac{Q\delta}{kA} = (T_{s1} - T_{s2}) \qquad (2)$$

Heat convected to cold fluid by wall surface is expressed as

$$Q = h_o A (T_{s1} - T_c)$$

$$\frac{Q}{h_o A} = (T_{s2} - T_c) \qquad (3)$$

Adding both sides of equations (1), (2) and (3), we get

$$Q\left(\frac{1}{h_i A} + \frac{\delta}{kA} + \frac{1}{h_o A}\right) = (T_h - T_c)$$

$$Q = \frac{(T_h - T_c)}{\left(\frac{1}{h_i A} + \frac{\delta}{kA} + \frac{1}{h_o A}\right)} \qquad (4)$$

We know that heat transfer in a heat exchanger can be expressed as

$Q = UA\,(T_h - T_c)$ (5)

Comparing equations (4) and (5), we can write

$$UA(T_h - T_c) = \frac{(T_h - T_c)}{\left(\frac{1}{h_i A} + \frac{\delta}{kA} + \frac{1}{h_o A}\right)}$$

$$\frac{1}{UA} = \left(\frac{1}{h_i A} + \frac{\delta}{kA} + \frac{1}{h_o A}\right) \qquad (6)$$

$$\frac{1}{U} = \left(\frac{1}{h_i} + \frac{\delta}{k} + \frac{1}{h_o}\right) \qquad (7)$$

Equation (7) represents overall heat transfer coefficient for aplane wall.

2. **<u>Double Pipe Heat Exchanger</u>**

Consider a double pipe heat exchanger of length 'L' in which hot fluid and cold fluid are flowing through inner and outer pipes respectively as shown in Figure 2.

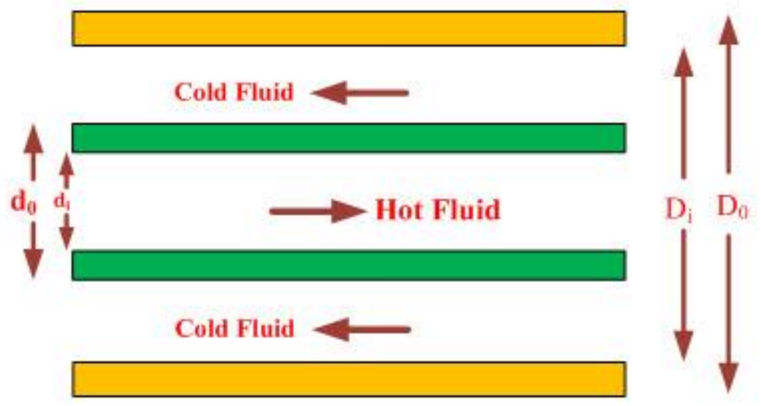



Figure 2

Let

$h_i$ and $h_o$ are convective heat transfer coefficients of hot and cold fluids with pipe surfaces respectively.

$d_i$ and $d_o$ are inside and outside diameters of inner pipe having thickness 'δ' and thermal conductivity 'k'.

$$\delta = \frac{d_o - d_i}{2} = r_o - r_i \tag{8}$$

$A_m$ is the mean peripheral area of inner pipe and is expressed as

$$d_m = \frac{d_o + d_i}{2} = \frac{A_o + A_i}{2} \tag{9}$$

$A_o = \Pi\, d_o L$ and $A_i = \Pi\, d_i L$

$D_i$ and $D_o$ are inside and outside diameters of outer pipe respectively.

Using equation (6), overall heat transfer coefficient, U for the arrangment can be expressed as

$$\frac{1}{UA_o} = \left(\frac{1}{h_i A_i} + \frac{\delta}{kA_m} + \frac{1}{h_o A_o}\right)$$

$$\frac{1}{U} = \left(\frac{A_o}{A_i}\frac{1}{h_i} + \frac{A_o}{A_m}\frac{\delta}{k} + \frac{1}{h_o}\right) \tag{10}$$

Substituting values of $A_i$ , $A_o$ and $A_m$ in equation (10), we get

$$\frac{1}{U}=\left(\frac{d_o}{d_i}\frac{1}{h_i}+\frac{d_o}{d_m}\frac{d_o-d_1}{2}\frac{1}{k}+\frac{1}{h_o}\right) \qquad (11)$$

Substituting value of $d_m$ from equation (9) in equation (11), we get

$$\frac{1}{U}=\left(\frac{d_o}{d_i}\frac{1}{h_i}+\frac{2d_o}{d_o+d_i}\frac{d_o-d_1}{2}\frac{1}{k}+\frac{1}{h_o}\right)$$

$$\frac{1}{U}=\left(\frac{d_o}{d_i}\frac{1}{h_i}+\frac{d_o-d_1}{d_o+d_i}\frac{d_o}{k}+\frac{1}{h_o}\right) \qquad (12)$$

Equation (12) represents overall heat transfer coefficient with reference to outer surface of inner pipe of a double pipe heat exchanger.

**Heat Transfer Analysis in Heat Exchanger**

Heat is transferred from hot fluid to cold fluid in heat exchanger and it involves the following steps

i) Heat lost by hot fluid is expressed as

$$Q_h = m_h c_{ph} (T_{hi} - T_{ho}) \qquad (13)$$

ii) Heat gained by cold fluid is expressed as

$$Q_c = m_c c_{pc} (T_{co} - T_{ci}) \qquad (14)$$

iii) Heat transfer in heat exchanger is expressed as

$$Q = U A (\Delta T)_m \qquad (15)$$

From energy balance considerations, heat transferred in heat exchanger is equal to heat lost by hot fluid which is gained by cold fluid.

$$Q = Q_h = Q_c \qquad (16)$$

Performance of a heat exchanger can be analyzed by using following methods

1. **Logarithmic Mean Temperature Difference or LMTD Method**
2. **Effectiveness of Number of Transfer Unit (NTU) Method**

**1. Logarithmic Mean Temperature Difference (LMTD) Method:**

LMTD method is used when temperatures of both the fluids at inlet and outlet of the heat exchanger are known. Performance analysis by this method is carried out by making following assumptions:

i) Overall heat transfer coefficient, U remains constant along the length of heat exchanger.

ii) Specific heats and mass flow rates of both the fluids remain constant.

iii) Heat exchanger is perfectly insulated and no loss of heat occurs.

iv) Axial conduction along the pipes is negligible.

**A. Parallel Flow Heat Exchanger:**

Consider a parallel flow heat exchanger of length 'L' as shown in Figure 3 and let $m_h$ and $m_c$ are mass flow rates of hot and cold fluids respectively.

$T_{hi}$ and $T_{ho}$ are temperatures of hot fluid at inlet and outlet of heat exchanger respectively.

$T_{ci}$ and $T_{co}$ are temperatures of cold fluid at inlet and outlet of heat exchanger respectively

$c_{ph}$ and $c_{pc}$ are specific heats of hot and cold fluids respectively.

$\theta_i$ and $\theta_o$ represent temperature difference between hot and cold fluids at inlet and outlet of heat exchanger respectively and are expressed as

$$\theta_i = T_{hi} - T_{ci} \text{ and } \theta o = T_{ho} - T_{co} \qquad (17)$$

$C_h$ and $C_c$ are heat capacity rates of hot and cold fluid respectively.

Heat capacity rate of a fluid is defined as amount of heat required to increase temperature of a fluid by 1 °C and is expressed as

$$C_h = m_h C_{ph,}, C_c = m_c C_{pc} \qquad (18)$$

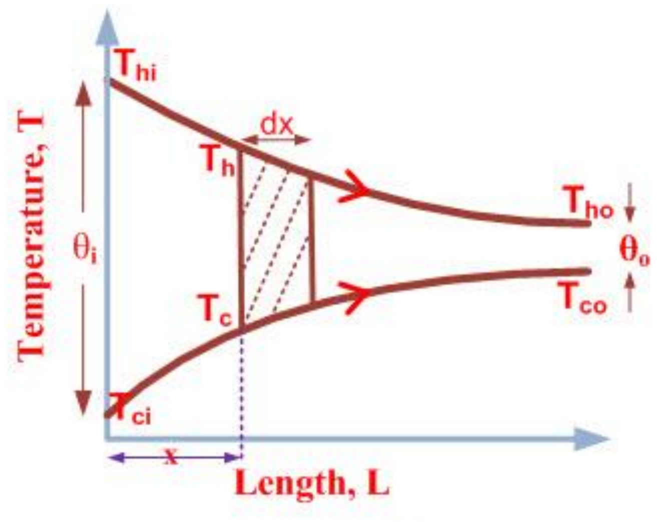



Figure 3

Consider an element of area dA and thickness dx at a distance 'x' from inlet of heat exchanger. Let $T_h$ and $T_c$ are temperatures of hot and cold fluid at inlet of the element and θ represents temperature difference of hot and cold fluid at inlet of the element.

$$\theta = T_h - T_c \qquad (19)$$

Let $dT_h$ and $dT_c$ represent change in temperature of hot and cold fluids respectively during the flow through the element. Let dθ be the difference in change in temperature of hot and cold fluids during flow through the element and is expressed as

$$d\theta = dT_h - dT_c \qquad (20)$$

During flow through the element, heat transferred is equal to heat lost by hot fluid and heat gained by cold fluid.

If dQ is the amount of heat transferred during flow of fluids through the element then

$dQ = UdA\,\theta = m_c\, c_{pc}\, dT_c = -\, m_h\, c_{ph}\, dTh$ (21)

(-ve sign if temperature of fluid decreases along the length of heat exchanger)

Using equation (18), equation (21) can be written as

$dQ = UdA\,\theta = C_c\, dT_c = -\, C_h\, dTh$ (22)

From equation (22), we can write

$$dT_c = \frac{UdA\theta}{C_c},\ dT_h = -\frac{UdA\theta}{C_h} \qquad (23)$$

Substituting the values of $dT_c$ and $dT_h$ from equation (23) in equation (20), we get

$$d\theta = -\frac{UdA\theta}{C_h} - \frac{UdA\theta}{C_c}$$

$$d\theta = -UdA\theta\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

$$\frac{d\theta}{\theta} = -UdA\left(\frac{1}{C_h} + \frac{1}{C_c}\right) \qquad (24)$$

Integrating equation (24) between limits $\theta_i$ and $\theta_o$

$$\int_{\theta_i}^{\theta_o} \frac{d\theta}{\theta} = -U\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\int_0^A dA$$

$$\log\left(\frac{\theta_o}{\theta_i}\right) = -UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

$$\log\left(\frac{\theta_i}{\theta_o}\right) = UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right) \qquad (25)$$

Heat lost by hot fluid during flow through heat exchanger is expressed as

$Q = m_h\, c_{ph}\, (T_{hi} - T_{ho}) = C_h\, (T_{hi} - T_{ho})$

Therefore,

$$C_h = \frac{Q}{(T_{hi} - T_{ho})} \qquad (26)$$

(26)

Heat gained by cold fluid during flow through heat exchanger is expressed as

$Q = m_c c_{pc} (T_{co} - T_{ci})$

Therefore,

$$C_c = \frac{Q}{(T_{co} - T_{ci})} \qquad (27)$$

(27)

Substituting values of $C_h$ and $C_c$ from equations (26) and (27) in equation (25), we get

$$\log\left(\frac{\theta_i}{\theta_o}\right) = UA\left(\frac{(T_{hi} - T_{ho})}{Q} + \frac{(T_{co} - T_{ci})}{Q}\right)$$

$$\log\left(\frac{\theta_i}{\theta_o}\right) = \frac{UA}{Q}\left[(T_{hi} - T_{ci}) - (T_{ho} - T_{co})\right] \qquad (28)$$

Using equation (17), equation (28) can be written as

$$\log\left(\frac{\theta_i}{\theta_o}\right) = \frac{UA}{Q}\left[(\theta_i) - (\theta_o)\right]$$

$$Q = UA\frac{\left[(\theta_i) - (\theta_o)\right]}{\log\left(\frac{\theta_i}{\theta_o}\right)} \qquad (29)$$

Comparing equations (29) and (15), we can write

$$UA(\Delta T)_m = UA\frac{\left[(\theta_i) - (\theta_o)\right]}{\log\left(\frac{\theta_i}{\theta_o}\right)}$$

$$(\Delta T)_m = \frac{\left[(\theta_i) - (\theta_o)\right]}{\log\left(\frac{\theta_i}{\theta_o}\right)} \qquad (30)$$

$$\frac{\log\left(\frac{\theta_i}{\theta_o}\right)}{\left[(\theta_i) - (\theta_o)\right]}$$

Equation 24.17 represents logarithmic mean temperature difference for a parallel flow heat exchanger.

**Lesson-27 Logarithmic mean temperature difference for counter and cross flow heat exchangers, Fouling or scaling of heat exchanger and Numerical problems**

Consider a counter flow heat exchanger of length 'L' as shown in Figure 1 and let $m_h$ and $m_c$ are mass flow rates of hot and cold fluids respectively.

$T_{hi}$ and $T_{ho}$ are temperatures of hot fluid at inlet and outlet of heat exchanger respectively.

$T_{ci}$ and $T_{co}$ are temperatures of cold fluid at inlet and outlet of heat exchanger respectively

$c_{ph}$ and $c_{pc}$ are specific heats of hot and cold fluids respectively.

$\theta_i$ and $\theta_o$ represent temperature difference between hot and cold fluids at inlet and outlet of heat exchanger respectively and are expressed as

$\boldsymbol{\theta_i = T_{hi} - T_{co}}$ **and** $\boldsymbol{\theta o = T_{ho} - T_{ci}}$ (1)

$C_{ph}$ and $C_{pc}$ are heat capacity rates of hot and cold fluid respectively.

Heat capacity rate of a fluid is defined as amount of heat required to increase temperature of a fluid by 1 °C and is expressed as

$C_{ph} = m_h C_{ph,}, C_{pc} = m_c C_{pc}$ (2)

Consider an element of area dA and thickness dx at a distance 'x' from inlet of heat exchanger. Let $T_h$ and $T_c$ are temperatures of hot and cold fluid at inlet of the element and $\theta$ represents temperature difference of hot and cold fluid at inlet of the element.

$\theta = T_h - T_c$ (3)

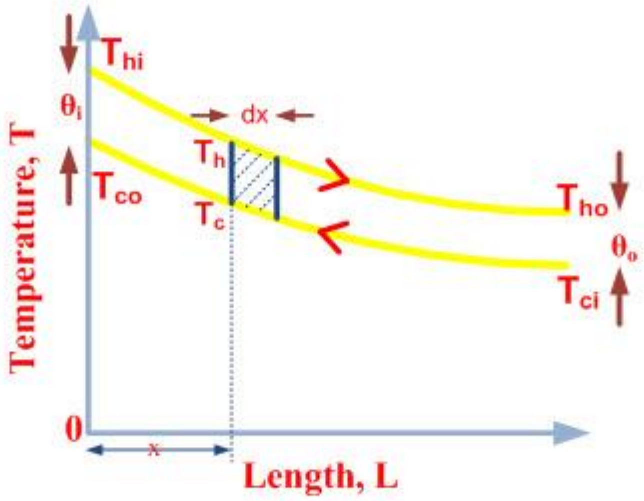



Figure 1

Let $dT_h$ and $dT_c$ represent change in temperature of hot and cold fluids respectively during the flow through the element. Temperature difference between hot and cold fluid at inlet of element is expressed as

$d\theta = dT_h - dT_c$ (4)

During flow through the element, heat transferred is equal to heat lost by hot fluid and heat gained by cold fluid.

If dQ is the amount of heat transferred during flow of fluids through the element then

$dQ = UdA\,\theta = -m_c\, c_{pc}\, dT_c = -\, m_h\, c_{ph}\, dTh$ (5)

(-ve sign if temperature of fluid decreases along the length of heat exchanger)

Using equation (2), equation (5) can be written as

$dQ = UdA\,\theta = -C_c\, dT_c = -\, C_h\, dTh$ (6)

From equation (6), we can write

$$dT_c = -\frac{UdA\theta}{C_c},\ dT_h = -\frac{UdA\theta}{C_h} \qquad (7)$$

Substituting the values of $dT_c$ and $dT_h$ from equation (7) in equation (4), we get

$$d\theta = -\frac{UdA\theta}{C_h} + \frac{UdA\theta}{C_c}$$

$$d\theta = UdA\theta\left(\frac{1}{C_c} - \frac{1}{C_h}\right)$$

$$\frac{d\theta}{\theta} = UdA\left(\frac{1}{C_c} - \frac{1}{C_h}\right) \qquad (8)$$

Integrating equation (8) between limits $\theta_i$ and $\theta_o$

$$\int_{\theta_i}^{\theta_o} \frac{d\theta}{\theta} = U\left(\frac{1}{C_c} - \frac{1}{C_h}\right)\int_0^A dA$$

$$\log\left(\frac{\theta_o}{\theta_i}\right) = UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right) \qquad (9)$$

Heat lost by hot fluid during flow through heat exchanger is expressed as

$Q = m_h\, c_{ph}\, (T_{hi} - T_{ho}) = C_h\, (T_{hi} - T_{ho})$

$$C_h = \frac{Q}{(T_{hi} - T_{ho})} \qquad (10)$$

Heat gained by hot fluid during flow through heat exchanger is expressed as

$Q = m_c c_{pc} (T_{co} - T_{ci}) = C_c (T_{co} - T_{ci})$

$$C_c = \frac{Q}{(T_{co} - T_{ci})} \qquad (11)$$

Substituting values of $C_h$ and $C_c$ from equations (10) and (11) in equation (9), we get

$$\log\left(\frac{\theta_o}{\theta_i}\right) = UA\left(\frac{(T_{co} - T_{ci})}{Q} + \frac{(T_{hi} - T_{ho})}{Q}\right)$$

$$\log\left(\frac{\theta_o}{\theta_i}\right) = \frac{UA}{Q}\left[(T_{ho} - T_{ci}) - (T_{hi} - T_{co})\right] \qquad (12)$$

Using equation (1), equation (12) can be written as

$$\log\left(\frac{\theta_o}{\theta_i}\right) = \frac{UA}{Q}\left[(\theta_o) - (\theta_i)\right]$$

$$Q = UA\frac{\left[(\theta_o) - (\theta_i)\right]}{\log\left(\frac{\theta_o}{\theta_i}\right)} \qquad (13)$$

We know that in a heat exchanger, heat transfer between hot and cold fluids can be expressed as

$Q = U A (\Delta T)_m$ (14)

Comparing equations (13) and (14), we can write

$$UA(\Delta T)_m = UA\frac{\left[(\theta_o) - (\theta_i)\right]}{\log\left(\frac{\theta_o}{\theta_i}\right)}$$

$$(\Delta T)_m = \frac{\left[(\theta_o) - (\theta_i)\right]}{\log\left(\frac{\theta_o}{\theta_i}\right)}$$

$$(\Delta T)_m = \frac{\left[(\theta_i) - (\theta_o)\right]}{\log\left(\frac{\theta_i}{\theta_o}\right)} \qquad (15)$$

Equation (15) represents logarithmic mean temperature difference for a counter flow heat exchanger.

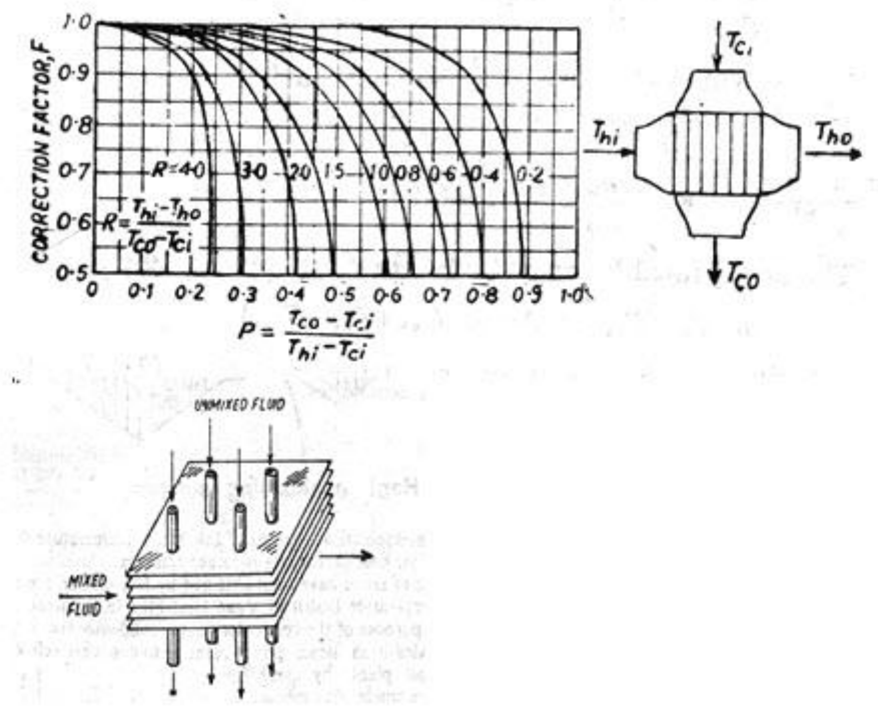
**C. Cross Flow Heat Exchanger**

In order to determine logarithmic mean temperature difference for a counter flow heat exchanger, following relationship is used.

$(LMTD)_{cross}$ = F X $(LMTD)_{counter}$

Where 'F' is a correction factor and is a function of geometry of heat exchanger and inlet as well as outlet temperatures of both the fluids. Values of correction factor for different arrangements of cross flow and multipass shell and tube heat exchangers are given in Figure 3.

**Fouling Factor**



**Lesson-28 Heat exchanger performance in terms of Capacity ratio, Effectiveness and Number of transfer units, Effectiveness for parallel flow heat exchanger**

## Effectiveness or Number of Transfer Units (NTU)

Logrithmic mean temperature difference method is very easy and simple to use for performance analysis of heat exchangers when inlet and outlet temperatures of hot and cold fluids are known or can be determined. However, if either temperature of the fluids is unknown or can not be determined, performance analysis by Logrithmic mean temperature difference method becomes cumbersome and complicated due to presence of logarithmic function. In such a case, effectiveness or number of transfer units (NTU) method is used for performance analysis of heat exchangers. Effectiveness or NTU method is based on the following concepts

- Capacity Ratio, C
- Effectiveness, ε
- Number of Transfer Units, NTU

**Capacity ratio (C):** It is defined as ratio of minimum to maximum capacity rate of fluids being used in a heat exchanger.

If $m_c\, c_{pc} < m_h\, c_{ph}$

$$\text{Capacity ratio, } C = \frac{m_c\, c_{pc}}{m_h\, c_{ph}} = \frac{c_{min}}{c_{max}} \qquad (1)$$

as $c_{min} = m_c\, c_{pc}$ and $c_{max} = m_h\, c_{ph}$

If $m_h\, c_{ph} < m_c\, c_{pc}$

$$\text{Or} \quad C = \frac{m_h\, c_{ph}}{m_c\, c_{pc}} = \frac{c_{min}}{c_{max}} \qquad (2)$$

as $c_{min} = m_h\, c_{ph}$ and $c_{max} = m_c\, c_{pc}$

The fluid with lower heat capacity rate will undergo greater change in temperature as compared to fluid with higher heat capacity rate.

**Effectiveness, ε:**

Effectiveness of a heat exchanger is defined as ratio of actual heat transferred to maximum possible heat that can be transferred.

$$\varepsilon = \frac{Q_{actual}}{Q_{max\ possible}} \qquad (3)$$

$Q_{actual}$ = Heat lost by hot fluid = Heat gained by cold fluid

$Q_{actual} = m_h c_{ph} (T_{hi} - T_{ho}) = m_c c_{pc} (T_{co} - T_{ci})$ (4)

Maximum possible heat transfer will be achieved if a fluid undergoes a change in temperature equal to maximum temperature difference available between hot and cold fluid.

Maximum Temperature difference that exists in a heat exchanger = $T_{hi} - T_{ci}$

Maximum possible heat transfer occurs when a fluid of smaller heat capacity rate undergoes a change in temperature equal to maximum available temperature difference.

$Q_{maxpossibe} = C_{min} (T_{hi} - T_{ci})$ (5)

Substituting values of $Q_{actual}$ and $Q_{max\ possible}$ from equations (4) and (5) in equation (3), we get

$$\varepsilon = \frac{m_c c_{pc} (T_{co} - T_{ci})}{C_{min} (T_{hi} - T_{ci})} = \frac{C_c (T_{co} - T_{ci})}{C_{min} (T_{hi} - T_{ci})} \qquad (7)$$

If $C_h < C_c$, then $C_h = C_{min}$, equation (6) can be expressed as

$$\varepsilon = \frac{C_{min} (T_{hi} - T_{ho})}{C_{min} (T_{hi} - T_{ci})} = \frac{(T_{hi} - T_{ho})}{(T_{hi} - T_{ci})} \qquad (8)$$

If $C_c < C_h$, then $C_c = C_{min}$, equation (7) can be expressed as

$$\varepsilon = \frac{C_{min} (T_{co} - T_{ci})}{C_{min} (T_{hi} - T_{ci})} = \frac{(T_{co} - T_{ci})}{(T_{hi} - T_{ci})} \qquad (9)$$

Effectiveness may also be defined as ratio of change in temperature of fluid with smaller capacity rate to maximum temperature difference existing in a heat exchanger.

**Number of Transfer Units, NTU:**

Number of transfer units is a dimensionless quantity and is an indicator of size of heat transferring areas of heat exchanger. Large value of NTU means larger heat transferring area. It is expressed as

$$NTU = \frac{UA}{C_{min}} \qquad (10)$$

**(a) Effectiveness for the Parallel Flow Heat Exchangers:**

From definition of effectiveness of a heat exchanger, we can write

$$\varepsilon = \frac{C_h (T_{hi} - T_{ho})}{C_{min} (T_{hi} - T_{ci})} = \frac{C_c (T_{co} - T_{ci})}{C_{min} (T_{hi} - T_{ci})} \qquad (11)$$

Heat transferred in a heat exchanger can be expressed as

$Q = C_h(T_{hi} - T_{ho}) = C_c(T_{co} - T_{ci})$

Consider an element of area dA and thickness dx at a distance 'x' from inlet of a parallel flow heat exchanger. Let $dT_h$ and $dT_c$ represent change in temperature of hot and cold fluids respectively during the flow through the element. Temperature difference between hot and cold fluid at inlet of element is expressed as

$d\theta = dT_h - dT_c$ (12)

If dQ is the amount of heat transferred during flow of fluids through the element then

$dQ = UdA\ \theta = C_c\, dT_c = -\ C_h\, dTh$ (13)

Substituting the values of $dT_h$ and $dT_c$ from equation (13) into the equation (12), we get

$$d\theta = -\frac{UdA\theta}{C_h} - \frac{UdA\theta}{C_c}$$

$$d\theta = -UdA\theta\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

$$\frac{d\theta}{\theta} = -UdA\left(\frac{1}{C_h} + \frac{1}{C_c}\right) \qquad (14)$$

Integrating equation (14) between limits $\theta_i$ and $\theta_o$, we get

$$\int_{\theta_i}^{\theta_o} \frac{d\theta}{\theta} = -U\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\int_0^A dA$$

$$\log_e\left(\frac{\theta_o}{\theta_i}\right) = -UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)$$

$$\left(\frac{\theta_o}{\theta_i}\right) = e^{-UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)} \qquad (15)$$

For a parallel heat exchanger,

$\theta_i = T_{hi} - T_{ci}$ and $\theta o = T_{ho} - T_{co}$

Equation (15) can be written as

$$\left(\frac{T_{ho} - T_{co}}{T_{hi} - T_{ci}}\right) = e^{-UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)} \qquad (16)$$

Using equation (4), values of $T_{ho}$ and $T_{co}$ can be written as

$$T_{ho} = T_{hi} - \frac{C_{min}}{C_h}(T_{hi} - T_{ci})\varepsilon \tag{17}$$

$$T_{co} = T_{ci} + \frac{C_{min}}{C_h}(T_{hi} - T_{ci})\varepsilon \tag{18}$$

Subtracting equation (18) from equation (17), we get

$$T_{ho} - T_{co} = (T_{hi} - T_{ci}) - C_{min}(T_{hi} - T_{ci})\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\varepsilon$$

$$T_{ho} - T_{co} = (T_{hi} - T_{ci})\left(1 - \varepsilon C_{min}\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\right)$$

$$\frac{(T_{ho} - T_{co})}{(T_{hi} - T_{ci})} = \left(1 - \varepsilon C_{min}\left(\frac{1}{C_h} + \frac{1}{C_c}\right)\right) \tag{19}$$

Using equation (19), equation (16) can be written as

$$1 - \varepsilon C_{min}\left(\frac{1}{C_h} + \frac{1}{C_c}\right) = e^{-UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)}$$

$$\varepsilon = \frac{1 - e^{-UA\left(\frac{1}{C_h} + \frac{1}{C_c}\right)}}{C_{min}\left(\frac{1}{C_h} + \frac{1}{C_c}\right)} \tag{20}$$

If $C_c < C_h$, then $C_c = C_{min}$ and $C_h = C_{max}$ equation (20) can be expressed as

$$\varepsilon = \frac{1 - e^{\frac{-UA}{C_{min}}\left(1 + \frac{C_{min}}{C_{max}}\right)}}{\left(1 + \frac{C_{min}}{C_{max}}\right)} \tag{21}$$

If $C_h < C_c$, then $C_h = C_{min}$ and $C_c = C_{max}$ equation (20) can be expressed as

$$\varepsilon = \frac{1 - e^{\frac{-UA}{C_{min}}\left(1 + \frac{C_{min}}{C_{max}}\right)}}{\left(1 + \frac{C_{min}}{C_{max}}\right)} \tag{22}$$

Using equations (2) and (10), equation (22) can be expressed as

$$\varepsilon = \frac{1 - e^{-NTU(1+C)}}{(1+C)} \tag{23}$$

The following two important cases are considered

i) Gas Turbine:

In case of a gas turbine $C = \frac{c_{min}}{c_{max}} \approx 1$ , Equation (23) can be expressed as

$$\varepsilon = \frac{1-e^{-2NTU}}{2} \qquad (24)$$

For a parallel flow heat exchanger, maximum value of effectiveness that can be achieved is 50% irrespective of values of its heat transferring area and overall heat transfer coefficient.

ii) Boiler and Condenser:

In case of a boiler and condenser, $C \frac{c_{min}}{c_{max}} \approx 0$ , Equation (23) can be expressed as

$$\varepsilon = 1-e^{-NTU} \qquad (25)$$

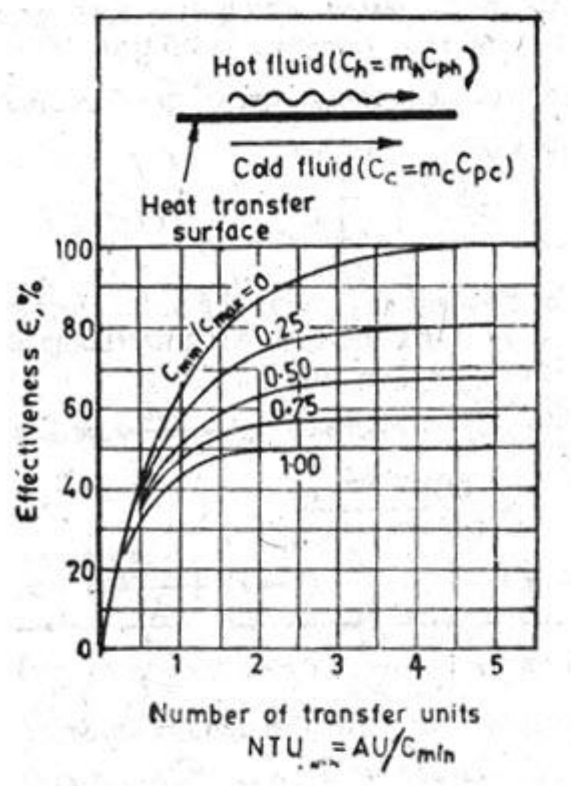



**Figure 1**

**Lesson-29 Effectiveness for counter flow heat exchanger and Numerical Problems**

**(b) Effectiveness for the Counter Flow Heat Exchangers:**

From definition of effectiveness of a heat exchanger, we can write

$$\varepsilon = \frac{C_h(T_{hi} - T_{ho})}{C_{min}(T_{hi} - T_{ci})} = \frac{C_c(T_{co} - T_{ci})}{C_{min}(T_{hi} - T_{ci})} \qquad (1)$$

Heat transferred in a heat exchanger can be expressed as

$Q = C_h(T_{hi} - T_{ho}) = C_c(T_{co} - T_{ci})$

Consider an element of area dA and thickness dx at a distance 'x' from inlet of a counter flow heat exchanger. Let $dT_h$ and $dT_c$ represent change in temperature of hot and cold fluids respectively during the flow through the element. Temperature difference between hot and cold fluid at inlet of element is expressed as

$d\theta = dT_h - dT_c$ (2)

If dQ is the amount of heat transferred during flow of fluids through the element then

$dQ = UdA\,\theta = -C_c\,dT_c = -C_h\,dTh$
(3)

Substituting the values of $dT_h$ and $dT_c$ from equation (4) into the equation (3), we get

$$d\theta = UdA\theta\left(\frac{1}{C_c} - \frac{1}{C_h}\right)$$

$$\frac{d\theta}{\theta} = UdA\left(\frac{1}{C_c} - \frac{1}{C_h}\right) \qquad (5)$$

Integrating equation (5) between limits $\theta_i$ and $\theta_o$

$$\int_{\theta_i}^{\theta_o} \frac{d\theta}{\theta} = U\left(\frac{1}{C_c} - \frac{1}{C_h}\right)\int_0^A dA$$

$$\log_e\left(\frac{\theta_o}{\theta_i}\right) = UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)$$

$$\left(\frac{\theta_o}{\theta_i}\right) = e^{UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)}$$

$$\left(\frac{\theta_i}{\theta_o}\right) = e^{-UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)} \qquad (6)$$

For a counter flow heat exchanger,

$\theta_i = T_{hi} - T_{co}$ and $\theta o = T_{ho} - T_{ci}$

Equation (6) can be written as

$$\left(\frac{T_{hi} - T_{co}}{T_{ho} - T_{ci}}\right) = e^{-UA\left(\frac{1}{C_c} - \frac{1}{C_h}\right)} \qquad (7)$$

Using equation (1), values of $T_{ho}$ and $T_{co}$ can be written as

$$T_{ho} = T_{hi} - \varepsilon \frac{C_{\min}}{C_h}(T_{hi} - T_{ci}) \qquad (8)$$

$$T_{co} = T_{ci} + \frac{C_{\min}}{C_h}(T_{hi} - T_{ci})\,\varepsilon \qquad (9)$$

Substituting the values of $T_{ho}$ and $T_{co}$ from equations (8) and (9) in equation (7), we get

$$\frac{\left(T_{hi}-T_{ci}-\varepsilon\frac{C_{min}}{C_c}\left(T_{hi}-T_{ci}\right)\right)}{\left(T_{hi}-T_{ci}-\varepsilon\frac{C_{min}}{C_h}\left(T_{hi}-T_{ci}\right)\right)}=e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)} \qquad (10)$$

$$\frac{\left(T_{hi}-T_{ci}\right)\left(1-\varepsilon\frac{C_{min}}{C_c}\right)}{\left(T_{hi}-T_{ci}\right)\left(1-\varepsilon\frac{C_{min}}{C_h}\right)}=e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)}$$

$$\frac{\left(1-\varepsilon\frac{C_{min}}{C_c}\right)}{\left(1-\varepsilon\frac{C_{min}}{C_h}\right)}=e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)}$$

$$1-\varepsilon\frac{C_{min}}{C_c}=e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)}-e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)}\varepsilon\frac{C_{min}}{C_h}$$

$$1-e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)}=\varepsilon\frac{C_{min}}{C_c}-e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)}\varepsilon\frac{C_{min}}{C_h}$$

$$\varepsilon\left(\frac{C_{min}}{C_c}-e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)}\frac{C_{min}}{C_h}\right)=1-e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)}$$

$$\varepsilon=\frac{1-e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)}}{C_{min}\left(\frac{1}{C_c}-\frac{1}{C_h}e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)}\right)} \qquad (11)$$

If $C_c < C_h$, then $C_c = C_{min}$ and $C_h = C_{max}$ equation (11) can be expressed as

$$\varepsilon=\frac{1-e^{-\frac{UA}{C_{min}}\left(1-\frac{C_{min}}{C_{max}}\right)}}{\frac{C_{min}}{C_{min}}\left(1-\frac{C_{min}}{C_{max}}e^{-UA\left(\frac{1}{C_c}-\frac{1}{C_h}\right)}\right)}$$

$$\varepsilon=\frac{1-e^{-\frac{UA}{C_{min}}\left(1-\frac{C_{min}}{C_{max}}\right)}}{\left(1-\frac{C_{min}}{C_{max}}e^{-\frac{UA}{C_{min}}\left(1-\frac{C_{min}}{C_{max}}\right)}\right)}$$

$$\varepsilon=\frac{1-e^{-NTU(1-C)}}{\left(1-Ce^{-NTU(1-C)}\right)} \qquad (12)$$

The following two important cases are considered

i) Boiler and Condenser:

In case of a boiler and condenser, $C = \frac{c_{min}}{c_{max}} \approx 0$, eqution (12) can be written as

$$\varepsilon = 1 - e^{-NTU}$$

ii) Gas Turbine:

In case of a gas turbine $C = \frac{c_{min}}{c_{max}} \approx 1$, Equation (12) becomes undeterminant and is solved as

Let $a = \frac{UA}{C_{min}}$ and $x = \frac{c_{min}}{c_{max}}$

So, equation (12) can be written as

$$\varepsilon = \frac{1 - e^{-a(1-x)}}{\left(1 - xe^{-a(1-x)}\right)} \quad \text{or} \quad \varepsilon = \frac{1 - \frac{1}{e^{a(1-x)}}}{\left(1 - \frac{x}{e^{a(1-x)}}\right)} = \frac{\frac{e^{a(1-x)} - 1}{e^{a(1-x)}}}{\frac{e^{a(1-x)} - x}{e^{a(1-x)}}} \quad \text{or}$$

$$\varepsilon = \frac{e^{a(1-x)} - 1}{\left(e^{a(1-x)} - x\right)} \qquad (13)$$

$$\Rightarrow \quad \varepsilon = \frac{1 + a(1-x) + \frac{a^2}{2!}(1-x)^2 + \frac{a^3}{3!}(1-x)^3 + \dots\dots\dots -1}{1 + a(1-x) + \frac{a^2}{2!}(1-x)^2 + \frac{a^3}{3!}(1-x)^3 + \dots\dots\dots - x}$$

$$\Rightarrow \quad \varepsilon = \frac{a(1-x) + \frac{a^2}{2!}(1-x)^2 + \frac{a^3}{3!}(1-x)^3 + \dots\dots\dots}{1 - x + a(1-x) + \frac{a^2}{2!}(1-x)^2 + \frac{a^3}{3!}(1-x)^3 + \dots\dots\dots}$$

$$\Rightarrow \quad \varepsilon = \frac{a(1-x) + \frac{a^2}{2!}(1-x)^2 + \frac{a^3}{3!}(1-x)^3 + \dots\dots\dots}{(1-x)\left(1 + a + \frac{a^2}{2!}(1-x) + \frac{a^3}{3!}(1-x) + \dots\dots\dots\right)}$$

$$\Rightarrow \quad \varepsilon = \frac{a + \frac{a^2}{2!}(1-x) + \frac{a^3}{3!}(1-x)^2 + \dots\dots\dots}{\left(1 + a + \frac{a^2}{2!}(1-x) + \frac{a^3}{3!}(1-x) + \dots\dots\dots\right)} \qquad (14)$$

If $x = \frac{c_{\min}}{c_{\max}} \rightarrow 1$ then (1-x)=0, equation (14) can be written as

$$\varepsilon = \frac{a}{a+1} \quad \text{or} \quad \varepsilon = \frac{NTU}{NTU+1}$$

If the value of NTU = 1 then ε = 0.5 and heat exchanger becomes 50% efficient. It should be noted that the equations developed for the heat exchanger effectiveness are independent of the terminal temperatures of the fluid but are explicit functions of the dimensionless ratios NTU and ε. This enables one to plot ε verses NTU for selected values of $C_{min}/C_{max}$.

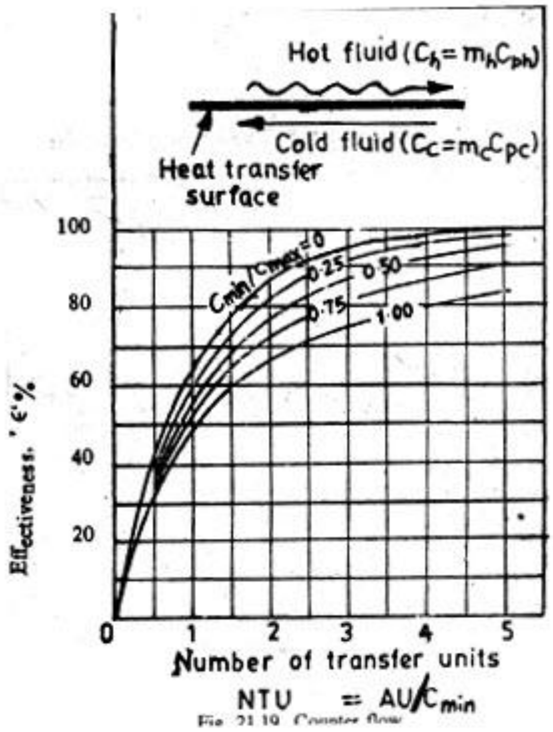



**Figure Counter Flow Heat exchanger**

In many heat exchangers, the fluids flow at right angles to each other are known as cross flow. The effectiveness relationships for four common flow arrangements, parallel, counter, cross flow one fluid mixed and other unmixed and cross flow both fluid mixed represented in form of graphs by Keys and London are shown in figures (3), (4), (5), and (6) respectively.

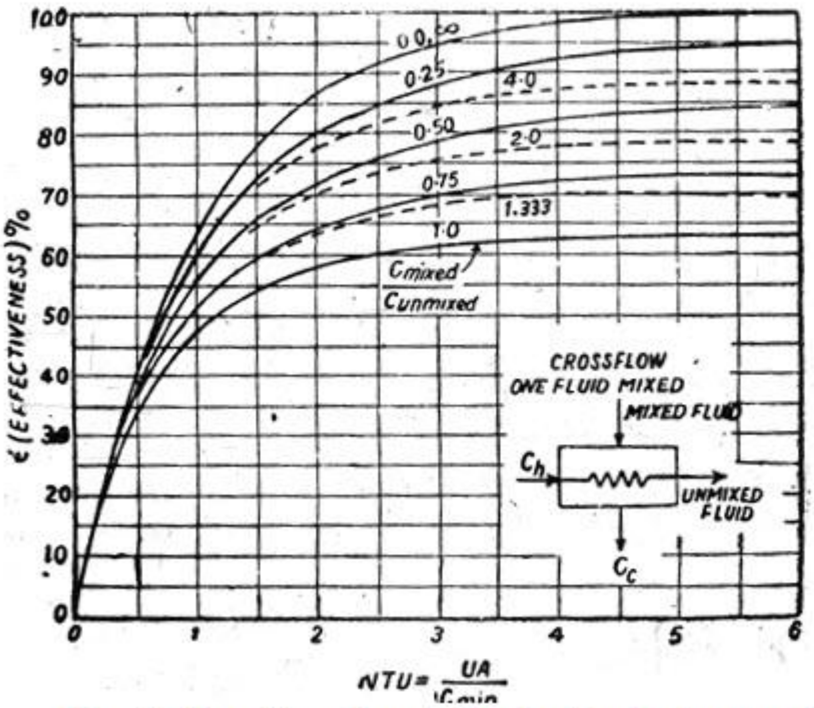



**Fig. 5: Cross flow. One fluid mixed and other unmixed.**

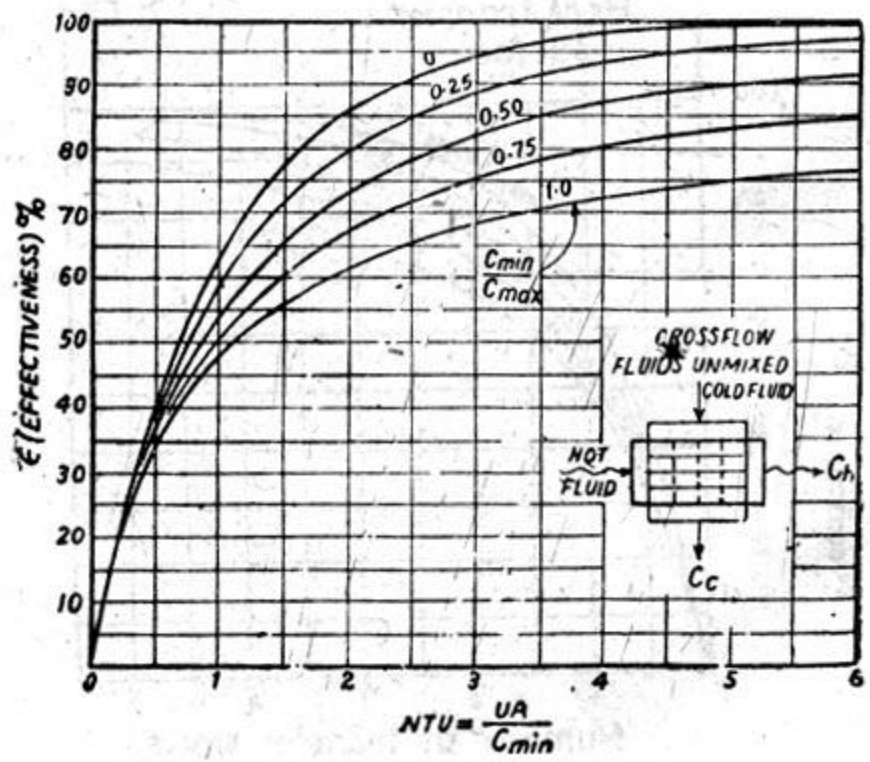



**Fig. 6: Cross flow both fluids mixed.**

**Example13.7 In a food processing plant, a brine solution is heated from -10°C to – 5.0°C in a double pipe parallel flow heat exchanger by water entering at 35°C and leaving at 20.5°C at the rate of 9 kg/min. Determine the heat exchanger area for an overall heat transfer coefficient of 860 W/m² K. For water $c_p$ = 4.186 10³ j/kgK.**

**Solution:**

Heat transfer from the water, $Q = m\, c_p\, \Delta t$

$$= 10 \times 4.186 \times 10^3 (30 - 20.5) = 439.530 \times 10^3 \text{J/min}$$

$$=7325.5.5 \text{ J/s}$$

Log-mean temperature difference, $\theta_m = \frac{\theta_1 - \theta_2}{\log_e(\theta_1/\theta_2)}$

For parallel flow arrangement (Figure 13.12):

$$\theta_1 = t_{h1} - t_{c1} = 30 - (-10) = 40°C$$

$$\theta_2 = t_{h2} - t_{c2} = 20.5 - (-5) = 25.5°C$$

The subscripts h and c represent the hot and cold fluids respectively:

$$\theta_m = \frac{40 - 25.5}{\log_e(40/25.5)} = 74.161°C$$

Heat exchange, $Q = U\, A\, \theta_m$

$\therefore$ Heating Surface area, $A = \frac{Q}{U\,\theta_m} = \frac{7325.5}{860 \times 74.161} = 0.11485\text{m}^2$

**Example13.8 A tubular heat exchanger is to be designed for cooling oil from a temperature of 80°C to 30°C by a large of stagnant water which may be assumed to remain constant at a temperature of 20°C. The heat transfer surface consists of 30 m long straight tube of 20 mm inside diameter. The oil (specific heat= 2.5 kj/kg k and specific gravity=0.8) flows through the cylindrical tube with an average velocity of 50 cm/s Calculate the overall heat transfer coefficient for the oil cooler .**

**Solution:**

The mass rate of flow of oil (hot fluid) through the heat transfer tube is

$$m_h = VA\rho = 0.5 \times \frac{\pi}{4}(0.02)^2 \times (0.8 \times 1000) = 0.1256 \text{ kg/s}$$

An energy balance on the hot fluid then gives the total energy transferred

$$Q = m_h c_h (t_{h1} - t_{h2}) = 0.1256 \times 2.5 \times (80 - 30) = 15.7 \text{ kJ/s}$$

The energy transferred is also given by

$$Q = U A \theta_m = U A \frac{\theta_1 - \theta_2}{\log_e(\theta_1/\theta_2)}$$

Since temperature of one of the fluid remains constant during the flow passage, it is immaterial whether the calculations are made for parallel flow or counter flow arrangement. Both arrangements would give the same values for log mean temperature difference and heating surface area for a specified load.

For parallel flow arrangement,

$$\theta_1 = t_{h1} - t_{c1} = 80 - (-20) = 60°\text{C}$$

$$\theta_2 = t_{h2} - t_{c2} = 30 - (-20) = 10°\text{C}$$

$$\theta_m = U A \frac{\theta_1 - \theta_2}{\log_e(\theta_1/\theta_2)} = \frac{60 - 10}{\log_e(60/10)} = 27.90°\text{C}$$

$$A = \pi\, d\, l = \pi \times 0.02 \times 30 = 1.884 \text{ m}^2$$

∴ Overall heat transfer coefficient for the oil cooler,

$$U = \frac{Q}{A\,\theta_m} = \frac{15.7}{1.884 \times 27.90} = 0.2987 \text{ kJ/m}^2\text{-s-deg}$$

**Example13.9 Exhaust gases ($c_p$=1.12 kJ/kg-deg) flowing through a tubular heat exchanger at the rate of 1200 kg/hr are cooled from 300°C to 20°C. The cooling is affected by water ($c_p$=4.18 kJ/kg-deg) that enters the system at 10°C at the rate of 1500 kg/hr. If the overall heat transfer coefficient is 500 kJ/m²-hr-deg, what heat exchanger area is required to handle the load for (a) parallel flow and (b) counter flow arrangement? .**

**Solution:**

The unknown exit temperature of the cooling water may be found from an energy balance on the two fluids, i.e.,

heat gained by water = heat lost by exhaust gases

$$m_c c_c (t_{c2} - t_{c1}) = m_h c_h (t_{h1} - t_{h2})$$

$$t_{c2} = t_{c1} + \frac{m_h c_h}{m_c c_c}(t_{h1} - t_{h2}) = 10 + \frac{1200 \times 1.12}{1500 \times 4.18}(300 - 20) \cong 70°C$$

From an energy balance on the hot fluid, the transfer rate is,

$$Q = m_h c_h (t_{h1} - t_{h2})$$

$$= 1200 \times 1.12(300 - 20) = 376320 \text{kJ/hr}$$

(a) **Parallel flow arrangement** (figure 13.12). The log mean temperature difference is,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e(\theta_1/\theta_2)}$$

Where $\theta_1 = t_{h1} - t_{c1} = 300 - 10 = 290°C$ and $\theta_2 = t_{h2} - t_{c2} = 20 - 10 = 10°C$

$$\therefore \theta_m = \frac{290 - 10}{\log_e(290/10)} = 191.466°C$$

Now, heat exchange $Q = U A \theta_m$

$\therefore$ Heating Surface area, $A = \frac{Q}{U \theta_m} = \frac{376320}{500 \times 191.466} = 3.930 m^2$

(b) **Counter flow arrangement** ( Figure 13.13). The log mean temperature difference is,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e(\theta_1/\theta_2)}$$

$\theta_1 = t_{h1} - t_{c1} = 300 - 80 = 220°C$

$\theta_2 = t_{h2} - t_{c2} = 120 - 10 = 110°C$

$$\therefore \theta_m = \frac{220 - 110}{\log_e(220/110)} = 365.412°C$$

Heat exchange $Q = U A \theta_m$

$\therefore$ Heating Surface area, $A = \frac{Q}{U \theta_m} = \frac{376320}{500 \times 365.412} = 2.059 m^2$

**Example13.14 A counter-flow concentric tube heat exchanger is used to cool the lubricating oil of a large industrial gas turbine engine. The oil flows through the tube at 0.16 kg/s ($c_p$=2.18 kJ/kgK), and the coolant water flows in the annulus in the opposite direction at a rate of 0.15 kg/s ($c_p$=4.18 kJ/kgK).The oil enters the coolant at 425 K and leaves at 345 K while the coolant enters at 285 K. How long must the tube be made to perform this duty if the heat transfer coefficient from oil to tube surface is 2250 W/m² K and from tube surface to water is 5650 W/m² K? The tube has a mean diameter of 12 mm and its wall presents negligible to heat transfer.**

**Solution:**

A heat balance over the exchanger gives:

Heat lost by oil = heat gained by water

$$m_0 c_0 (t_{01} - t_{02}) = m_w c_w (t_{w2} - t_{w1})$$

The subscripts 0 and w refer to oil and water respectively. Inserting the appropriate values:

$$0.16 \times 2.18(425 - 345) = 0.15 \times 4.18(t_{w2} - 285)$$

∴ Outlet temperature of water;

$$t_w = 285 + \frac{0.16 \times 2.18 \times 80}{0.15 \times 4.18} = 329.5 \text{ K}$$

From an energy balance on the oil, the heat transfer is given by

$$Q = m_0 c_0 (t_{01} - t_{02})$$

$$= 0.16 \times 2.18(425 - 345) = 27.904 \text{ kJ/s} = 27.904 \times 10^5 \text{J/s}$$

For counter current operation of the exchanger (Fig.13.13),

$$\theta_1 = t_{01} - t_{w2} = 425 - 329.5 = 95.5 \text{ K}$$

$$\theta_2 = t_{02} - t_{w1} = 345 - 285 = 60 \text{ K}$$

∴ Log-mean temperature difference,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e(\theta_1/\theta_2)} = \frac{95.5 - 60}{\log_e(95.5/60)} = 72.79 \text{ K}$$

Let U be overall heat transfer coefficient between oil and water;

$$\frac{1}{U} = \frac{1}{h_o} + \frac{1}{h_w}$$

$$U = \frac{h_o h_w}{h_o + h_w} = \frac{2250 \times 5650}{2250 + 5650} = 1609.18\ W/m^2K$$

Heat exchange $Q = U A \theta_m$

$$\therefore \text{Heating Surface area, } A = \frac{Q}{U\,\theta_m} = \frac{33.136 \times 10^3}{1609.18 \times 7279} = 0.2829\ m^2$$

The surface area also equals $\pi$ d l where d and l represent the tube and length respectively.

$\therefore$ Required length of the tube is

$$l = \frac{0.2829}{\pi \times 12.5 \times 10^{-3}} = 7.21\ m$$

**Example13.15 A one-shell, two-tube pass heat exchanger having 4000 thin wall brass tubes of 20 mm diameter has been installed in a stream power plant with a heat load of 2.310$^8$ W. The stream condenses at 60°C and the cooling water enters the tubes at 10°C at the rate of 3000kg/s. Calculate the overall heat transfer coefficient, the tube length per pass, and the outer surfaces of the tubes as 15500 W/m$^2$ K and the latent heat of stream as 2380 kJ/kg.**

**Further presume the following fluid properties:**

**c = 4180 j/kgK, μ=85510 $^{-6}$ Ns/m$^2$, k = 0.613 W/mK and Pr = 5.83**

**Solution:**

For thin walled tubes, the overall heat transfer coefficient is given by:

$$\frac{1}{U} = \frac{1}{h_o} + \frac{1}{h_w} \quad \text{Or} \quad U = \frac{h_o h_w}{h_o + h_w}$$

The heat transfer coefficient for condensation on the outer surfaces of the tubes is stated as $h_0$ =15500 W/m$^2$ and the coefficient on the inner surfaces $h_i$ can be calculated from the internal flow situation as follows:

$$Re = \frac{V\,d\,\rho}{\mu} \quad \frac{4m}{\pi\,d\,\mu}$$

where m is the mass flow rate through each tube=1 kg/s

$$Re = \frac{4 \times 1}{\pi \times 0.02 \times (855 \times 10^{-6})} = 74496$$

Obviously the flow is turbulent and therefore the following correlation applies:

$$N_u = 0.023(Re)^{0.8}\,(Pr)^{0.4}$$

$$= 0.023 \times (74496)^{0.8} \times (5.83)^{0.4} = 367.6$$

$$\therefore \quad h_i = \frac{Nu \times k}{d} = \frac{367.6 \times 0.613}{0.02} = 11267\ W/m^2k$$

And so,

$$U = \frac{11267 \times 15500}{11267 \times 15500} = 6524\ W/m^2k$$

The outlet temperature of cooling water can be obtained from the energy balance. That is

$$Q = m_c c_c (t_{c2} - t_{c1})$$

$$2.3 \times 10^8 = 3000 \times 4180 \times (t_{c2} - 10)$$

$$\therefore\ t_{c2} = \frac{2.3 \times 10^8}{3000 \times 4180} + 10 = (t_{c2} - 10)$$

Log mean temperature difference,

$$\theta_m = \frac{\theta_1 - \theta_2}{\log_e(\theta_1/\theta_2)}$$

Where; $\theta_1 = t_{h1} - t_{c1} = 60 - 10 = 50°C$

$\theta_2 = t_{h2} - t_{c2} = 60 - 38.34 = 21.66°C$

$$\therefore \theta_m = \frac{50 - 21.66}{\log_e(50/21.66)} = \frac{28.34}{.363} = 19.40°C$$

Heat exchanger, $Q = U A \theta_m$

$\therefore$ Heating Surface area, $A = \frac{Q}{U\,\theta_m} = \frac{2.3\times10^8}{6524\times19.40} = 1817\ m^2$

For a two pass heat exchanger, the surface area also equals, N× $2\pi$ d l, where N is the number of tubes, d and l represent the tube diameter and length respectively,

$\therefore$ l(length of tube per pass) $= \frac{1817}{3000\times2\times\pi\times0.02} = 4.82\ m$

The mass flow rate of condensation is obtained from, $Q = m \times h_{fg}$

$\therefore$ mass flow of condensation, $m = \frac{2.3\times10^8}{2380\times1000} = 96.64\ kg/s$

**Example13.16 A heat exchanger is to be designed to condense 9 kg/s of an organic liquid ($t_{sat}$=80°C; $h_{fg}$=600 kj/kg) with cooling water at 16°C and at a flow rate of 55 kg/s. The overall heat transfer coefficient is 480 W/m²-deg. Calculate:**

**a) the number of tubes required. The tubes are to be of 25 mm outer diameter, 2 mm thickness and 5 m length.**

**b) the number of tube passes. The velocity of the cooling water is not to exceed 2 m/s**

**Solution:**

From energy balance,

Heat lost by vapour = heat gained by water

$M_h h_{fg} = m_c c_c (t_{c2} - t_{c1})$

$9 \times 600 = 55 \times 4.186(t_{c2} - 16)$

∴ Outlet temperature of water,

$$t_{c2} = \frac{9 \times 600}{55 \times 4.186} + 16 = 39.45°C$$

Logarithmic mean temperature difference, $\theta_m = \frac{\theta_1 - \theta_2}{\log_e(\theta_1/\theta_2)}$

For a condensing vapour, the temperature remains constant through out the flow passage. That is $t_{h1} - t_{c1} = 80°C$

$\theta_1 = t_{h1} - t_{c1} = 80 - 16 = 64°C$

$\theta_2 = t_{h2} - t_{c2} = 80 - 39.45 = 40.55°C$

$$\therefore \theta_m = \frac{64 - 40.55}{\log_e(64/40.55)} = 118.32°C$$

Heat transfer rate is given by,

$$Q = m_h h_g \, U A \theta_m$$

$$= U(\pi d_0 l N) \times \theta_m$$

Or $\quad 9 \times (600 \times 10^3) = 480 \times (\pi \times 0.025 \times 5 \times N) \times 118.32$

∴ Number of tubes, N = 478

Let n be the number of tubes in each pass. Then mass of cold water passing through each pass,

$$m_c = \left(\frac{\pi}{4} d_i^2 \times V \times P\right) n$$

$$\therefore 55 = \frac{\pi}{4}(0.021)^2 \times 2 \times 1000 \times n \quad ; \quad n = 86.66$$

Number of passes = $\frac{478}{86.66} = 5.51 = 6$

**Lesson-30 Numerical Problems related to heat exchanger performance**

**Example13.20 A counter flow heat exchanger is used to cool 2200 kg/hr of oil ($c_p$=2.5 kJ/kgK), from 100°C to 35°C by the use of water entering at 17°C. If the overall heat transfer coefficient is expected to be 1.5 kW/m²k, make calculations for the water flow rate, the surface area required and the effectiveness of heat exchanger. Presume that the exit temperature of water is not to exceed 85°C. Use NTU-effectiveness approach.**

**Solution:**

The mass flow rate of water can be determined from an energy balance on the two fluids, i.e.,

Heat lost by oil (hot fluid) = heat gained by water (coolant)

$$m_h c_h(t_{h1} - t_{h2}) = m_c c_c(t_{c2} - t_{c1})$$

$$2200 \times 2.5(100 - 35) = m_c \times 4.18(85 - 17)$$

∴ mass flow rate of coolant (water) $m_c = \frac{2200\times2.5\times(100-35)}{4.18\times(85-17)} = 1257.74$ kg/hr

(b) Thermal capacity of the water stream (coolant)

$$C_c = m_c c_c = 1257.74 \times 4.18 = 5257.35 \text{ kj/hrK}$$

Thermal capacity of the oil stream (hot fluid)

$$C_h = m_h c_h = 2200 \times 2.5 = 5500 \text{ kj/hrK}$$

Obviously $C_{min} = 5500$ kJ/hrK and $C_{max} = 5257.35$ kJ/hrK

Capacity ratio $C = \frac{C_{min}}{C_{max}} = \frac{5500}{5257.35} = 1.046$

When hot fluid has the minimum heat capacity, then

Effectiveness $\in = \frac{(t_{h1}-t_{h2})}{(t_{h1}-t_{c1})} = \frac{100-35}{100-17} = 0.783$

The number of transfer units (NTU) can be computed from the following expression for effectiveness of a counter flow exchanger

$$\in = \frac{1 - \exp[-NTU(1 - C)]}{1 - C\exp[-NTU(1 - C)]}$$

Rearranging

$$\frac{\epsilon-1}{\epsilon\,C-1} = \exp\left[-NTU\,(1-C)\right]$$

$$\frac{0.783-1}{0.783\times1.046-1} = \exp\left[-NTU\,(1-1.046)\right]$$

Or $\log_e \frac{0.217}{0.180} = 0.133\,NTU \quad \text{or } NTU = 3.84$

Alternatively using the parameters C = 0.867 and ∈= 0.833 with Fig. 13.16, we get; NTU= 3.84

But $NTU = \frac{UA}{C_{min}}$

Therefore the heat transfer area is

$$A = \frac{NTU\times C_{min}}{U} = \frac{3.84\times5000}{1.5\times3600} = 3.55m^2$$

**Example13.23A home air-conditioning system uses a counter flow heat exchanger to cool 0.8 kg/s of air from 45°C to 15°C. The cooling is accomplished by a stream of cooling water that enters the system with 0.5 kg/s flow rate and 8°C temperature. If the overall heat transfer coefficient is 35W/m²K, what heat exchanger area is required? If the same air flow rate is maintained while the water flow rate is reduced to half, how much will be the percentage reduction in heat transfer? Use effectiveness-NTU approach.**

**Solution:**

Thermal capacity rates for the hot fluid (air) and cold fluid (water) are:

$$C_h = m_h c_h = 0.8\times1005 = 804\ kj/hrK$$

$$C_c = m_c c_c = 0.75\times4186 = 3139.5\ kj/hrK$$

Obviously $C_{max}$ = 3139.5 W/K and $C_{min}$ = 804 W/K

Capacity ratio, $C = \frac{C_{min}}{C_{max}} = \frac{804}{3139.5} = 0.256$

Effectiveness, $\in = \frac{(t_{h1}-t_{h2})}{(t_{h1}-t_{c1})} = \frac{45-15}{45-8} = 0.81$

The effectiveness for a counter flow arrangement is,

$$\in = \frac{1-\exp\left[-NTU\,(1-C)\right]}{1-C\exp\left[-NTU\,(1-C)\right]}$$

Rearranging: $\frac{\epsilon-1}{\epsilon\,C-1} = \exp\left[-NTU\,(1-C)\right]$

$$\frac{0.81-1}{0.81\times0.256-1} = \exp\left[-NTU\,(1-0.256)\right]$$

Or $\log_e\frac{0.19}{0.793} = -0.744\,NTU \quad \text{or } NTU = 1.92$

Therefore, the required transfer area is,

$$A = \frac{NTU\times C_{min}}{U} = \frac{1.92\times804}{35} = 44.10\ m^2$$

(b) Since the water flow rate is reduced to half,

$$m_c = \frac{1}{2} \times 0.75 = 0.375 \text{ kg/s}$$

$$C_c = m_c c_c = 0.375 \times 4186 = 1569.75 \text{ W/K}$$

$$C_h = m_h c_c = 804 \text{ W/K} \text{ as it remains same}$$

$$C = \frac{C_{min}}{C_{max}} = \frac{804}{1569.75} = 0.5122$$

Transfer units $NTU = \frac{UA}{C_{min}}$

Since all the parameters in the above expression for NTU remain same, NTU will be same as in the original situation. That is

NTU =1.92

Invoking the effectiveness relation for counter flow heat exchanger

$$\epsilon = \frac{1 - \exp[-NTU(1-C)]}{1 - C\exp[-NTU(1-C)]}$$

$$= \frac{1-\exp[-1.92(1-0.5122)]}{1-0.5122\exp[-1.92(1-0.5122)]}$$

$$= \frac{1-\exp[-0.9366)]}{1-0.5122\exp[-0.9366]} = \frac{1-0.3919}{1-0.5122\times 0.3969} = 0.76$$

Also, $\epsilon = \frac{(t_{h1}-t_{h2})}{(t_{h1}-t_{c1})}$; $0.76 = \frac{\Delta t_h}{45-8}$; $\Delta t_h = 28.12°$

The rate of heat flow is proportional to $\Delta T_h$. Hence percentage change in heat transfer is

$$= \frac{(\Delta t_h)_{old} - (\Delta t_h)_{new}}{(\Delta t_h)_{oil}} = \frac{30 - 28.12}{30} = 0.0627 \text{ or } 6.27\%$$

**Example13.26 The Engine oil at 150°C is cooled at 80°C in a parallel flow heat exchanger by water entering at 25°C and leaving at 60°C. Estimate the exchanger effectiveness and the number of transfer units. If the fluid flow rates and the inlet conditions remain unchanged, work out the lowest temperature to which the oil may be cooled by increasing length of the exchanger.**

**Solution:**

From energy balance on the hot (oil) and cold (water) fluids,

$$m_c c_c(t_{c2} - t_{c1}) = m_h c_h(t_{h1} - t_{h2})$$

$$\frac{m_h c_h}{m_c c_c} = \frac{t_{c2} - t_{c1}}{t_{h1} - t_{h2}} = \frac{60 - 25}{150 - 80} = 0.5$$

Apparently, the hot fluid has the minimum thermal capacity and $C_{min}/C_{max}$ =0.5

$$\text{Effectiveness, } \epsilon = \frac{C_h(t_{h1} - t_{h2})}{C_{min}(t_{h1} - t_{c1})} = \frac{(t_{h1} - t_{h2})}{(t_{h1} - t_{c1})} = \frac{150 - 80}{150 - 25} = 0.56$$

In terms of capacity ratio and number of transfer units,

$$\epsilon = \frac{1 - \exp[-NTU(1 + C)]}{1 + C}$$

$$0.56 = \frac{1 - \exp[-NTU(1 + 0.5)]}{1 + 0.5} = \frac{1 - \exp[-NTU(-1.5NTU)]}{1.5}$$

$$\exp[-1.5NTU] = 1 - 1.5 \times 0.56 = 0.16$$

∴ Number of transfer units, NTU =1.221

(b) The exit temperature of oil would be minimum corresponding to the situation when the exchanger is increased infinity or NTU→ ∞.

$$\therefore \epsilon = \frac{1 - \exp[-NTU(1 + C)]}{1 + C} = \frac{1}{1 + C} = \frac{1}{1 + 0.5} = 0.667$$

Also,

$$\epsilon = \frac{(t_{h1} - t_{h2})}{(t_{h1} - t_{c1})} \quad \text{or} \quad \frac{105 - t_{h2}}{105 - 25}$$

Therefore, the minimum possible exit temperature of the oil is

$$t_{h2} = 150 - 0.667(150 - 25) = 66.62°C$$

**Example21.5 Fuel oil at the rate of 1.3 kg/s is heated passing through the annulus of a computer flow double pipe heat exchanger from 20°C to 30°C by using hot water available from the engine at 70°C. The water flows through a copper tube (OD=2.13 cm and ID=1.86 cm) with a velocity of 0.76 m/s. The oil passes through the annulus formed by inner copper tube and outer steel pipe (OD=3.34cm and ID=3 cm).**

**$F_w$ (Fouling factor water side)=0.0004 m² C/W**

**$F_0$ (Fouling factor oil side)=0.0009 m² C/W**

**Take the following properties of water and oil**

| | Water | Oil |
|---|---|---|
| (kg/m³) | 980 | 850 |
| C,(kj/kg-°C) | 4.2 | 2 |
| K (W/m-C°) | 0.7 | 0.038 |
| V(m²/s) | 4.2 | 7.5 |

**Neglect the resistance of copper tube.**

**Solution:**

$$m_w = \rho U_m A_c = 980 \times 0.76 \times \frac{\pi}{4}\left(\frac{1.86}{100}\right)^2 = 0.199\text{kg/s}$$

Heat lost by water = Heat gained by oil

$$m_w C_{pw}(T_{hi} - T_{h0}) = m_0 C_{p0}(T_{00} - T_{0i})$$

$$0.2 \times 4.2 \times 10^3 (71 - T_{h0}) = 1.3 \times 2 \times 10^3 \times (30 - 20)$$

$$T_{h0} = 70 - \frac{1.3 \times 2 \times 10}{0.2 \times 4.2} = 70 - 30.95 = 39.05°\text{C}$$

$$\text{LMTD} = \frac{\theta_i - \theta_0}{\log_e\left(\frac{\theta_i}{\theta_0}\right)} = \frac{(70-30)-(39.05-20)}{\log_e\frac{(70-30)}{(39.05-20)}} = 65.02°\text{C}$$

**Water side**

$$Re_W = \frac{d_i v}{v} = \frac{0.0186 \times 0.76}{5 \times 10^{-7}} = 28.27 \times 10^3$$

$$Pr_w = \frac{\rho v C_p}{k} = \frac{980 \times 5 \times 10^{-7} \times 4.2 \times 10^3}{0.7} = 2.94$$

The average heat transfer coefficient of water side is given by

$$Nu_a = \frac{h_i d_i}{K} = 0.023(Re_w)^{0.8}(Pr_w)^{0.3}$$

$$\therefore h_i = 0.023(33.82 \times 10^3)^{0.8}(2.94)^{0.3} \times \frac{0.7}{0.0186} = 5025.23\ \text{W/m}^2 - °\text{C}$$

**Fuel Oil side**

$$D_e\,(\text{equivalent diameter}) = 3 - 2.13 = 0.87\ \text{cm}$$

$$Re_0 = \frac{D\,.\,U_m\,\rho}{\mu} = \frac{D_e}{\mu}\,.\left(\frac{G}{A}\right) = \frac{D_e}{\nu\rho}\left(\frac{G}{A}\right)$$

Where G is the mass flow per second and A is the area of annulus through which oil flows.

$$\therefore Re_0 = \frac{0.87\times10^{-2}}{7.43\times10^{-4}\times854}\times\frac{1.3}{\pi(0.03^2-0.0213^2)} = 5081.8$$

$$Pr_0 = \frac{C_p\rho\nu}{K} = \frac{2\times10^3\times850\times7.5\times10^{-6}}{0.038} =$$

The average heat transfer coefficient referred to outside surface of copper tube is given by

$$\frac{1}{U} = \left(\frac{d_0}{d_i}\times\frac{1}{h_i}+\frac{d_0}{d_i}f_w\right)+\left(\frac{1}{h_0}+f_0\right)$$

$$\frac{1}{U} = \frac{d_0}{d_i}\left(\frac{1}{h_i}+f_w\right)+\left(\frac{1}{h_0}+f_0\right)$$

$$= \frac{2.13}{1.86}\left(\frac{1}{5025.23}+0.0004\right)+\left(\frac{1}{1750}+0.0009\right)$$

$$= 0.00068 + 0.00147 = 0.00215$$

$$\therefore U = 480\ \text{W/m}^2-{}^\circ\text{C}.$$

$$Q = UA(LMTD) = U\times(\pi d_0 L)\times(LMTD)$$

$$Q = \text{Heat gained by oil}$$

$$= 1.1\times1.884\times10^3(20-10) = 20700\text{W}$$

$$\therefore 20700 = 480\left(\pi\times\frac{2.13}{100}\times L\right)\times 43.5$$

$$L = \frac{20700\times100}{480\times\pi\times2.13\times43.5} = 15\ \text{meters.}$$

**Example21.8 A heat exchanger has 17.5 m² area available for heat transfer. It is used for cooling oil at 200°C by using water available at 20°C. The mass flow and specific heat of oil are 10000 kg/hr and 1.9 kJ/kg K and the mass flow and specific heat of water are 3000kg/hr. and 4.187 kJ/kg K. if the overall heat transfer coefficient is 300 W/m²-K, estimate the outlet temperatures of oil and water for parallel flow and counter flow arrangements (a) by using LMTD method and (b) NTU method**

**Solution:**

$$C_r = m_r \;.\; C_{ph} = \frac{10{,}000}{3600} \times 1900 = 5280 = C_{max}$$

$$C_0 = m_0 \;.\; C_{ph} = \frac{3000}{3600} \times 4187 = 3480 = C_{min}$$

Heat lost by hot fluid = Heat gained by cold fluid

$$m_r C_{pr}(T_{hi} - T_{h0}) = m_0 C_{p0}(T_{00} - T_{0i})$$

$$5280(200 - T_{h0}) = 3480(T_{00} - 20)$$

$$T_{00} = 322.6 - 1.513 T_{h0} \qquad \text{....(a)}$$

This equation is valid for parallel as well as for counter flow.

**A LMTD-Method**

**(i) Parallel Flow**

$$Q = m_r\, C_{pr}(T_{hi} - T_{h0}) = UA\left[\frac{\theta_i - \theta_o}{\log_e\left(\frac{\theta_i}{\theta_o}\right)}\right]$$

$$\therefore \quad 5280(200 - T_{h0}) = 300 \times 17.5 \left[\frac{180 - (T_{hi} - T_{00})}{\log_e\left(\frac{180}{T_{hi} - T_{00}}\right)}\right]$$

Substituting the value of $T_{00}$ from equation (a) into the above equation and simplifying.

$$\log_e\left[\frac{180}{2.513\, T_{h0} - 322.6}\right] = 1.01 \times 2.513\left(\frac{200 - T_{h0}}{200 - T_{00}}\right) = 2.54$$

$$\therefore \quad \left[\frac{180}{2.513\, T_{h0} - 322.6}\right] = (e)^{2.54} = 12.68$$

$$\therefore \; T_{h0} \;=\; 134°C$$

$$\therefore \quad T_{00} = 322.6 - (1.513 \times 134) = 120°C$$

**(ii) Counter Flow**

$$Q = m_h\, C_{ph}(T_{hi} - T_{r0}) = UA\left[\frac{\theta_i - \theta_0}{\log_e\left(\frac{\theta_i}{\theta_0}\right)}\right]$$

$$\therefore \quad 5280(200 - T_{h0}) = 300 \times 17.5\left[\frac{(200 - T_{00}) - (T_{h0} - 20)}{\log_e\left(\frac{200 - T_{00}}{T_{h0} - 20}\right)}\right]$$

Again substituting for $T_{00}$ from equation (a) in the above equation and simplifying, we get

$$\log_e\left[\frac{1.513T_{h0} - 122.6}{T_{h0} - 20}\right] = \frac{0.513(T_{h0} - 200)}{0.986(200 - T_{h0})} = -0.52$$

$$\therefore \frac{1.513T_{h0} - 122.6}{T_{h0} - 20} = (e)^{-0.52} = 0.595$$

$$\therefore \; T_{h0} = 120°\text{C}$$

$$\therefore \quad T_{00} = 322.6 - 1.513\,T_{h0} = 322.6 - 1.513 \times 120 = 140°\text{C}$$

**B NTU Method**

$$\frac{C_{min}}{C_{max}} = \frac{3480}{5280} = 0.66 = P$$

$$NTU = \frac{UA}{C_{min}} = \frac{300 \times 17.5}{3480} = 1.5$$

The value of /NTU is also independent of flow direction.

**(i) Parallel Flow**

$$\epsilon = \frac{\left[1 - (e)^{-NTU(1+P)}\right]}{(1+P)} = \frac{\left[1 - (e)^{-1.5 \times 1.66}\right]}{(1+0.66)} = 0.55$$

$$\epsilon = \frac{T_{00} - T_{0i}}{T_{hi} - T_{0i}}$$

$$\therefore \quad 0.55 = \frac{T_{00} - 20}{200 - 20}$$

$$\therefore \quad T_{00} = 20 + 0.55(200 - 20) = 120°\text{C}$$

Using equation (a)

$120 = 322.6\text{-}1.513\ T_{r0}$

$T_{h0} = 134°\text{C}$

**(ii) Counter Flow**

$$\in = \frac{1-(e)^{-NTU(1-P)}}{1-P(e)^{-NTU(1-P)}}$$

$$= \frac{1-(e)^{-1.5\times0.34}}{1-0.66(e)^{-1.5\times0.34}} = 0.67$$

$$\in = \frac{T_{00}-T_{0i}}{T_{hi}-T_{0i}}$$

$$\therefore T_{00} = T_{0i} + \in (T_{hi} - T_{0i})$$

$$20+0.67(200-20) = 140°C$$

Again using the equation (a)

$$140 = 322.6-1.513T_{h0}$$

$$\therefore T_{h0} = 120°C$$

Note. When NTU and () Values are known, then we can directly read the value of  from the graphics instead of calculation.

**Module 5. Mass Transfer**

## Lesson-31 Introduction, Fick's law of Diffusion, Mass Transfer Coefficients

Mass transfer plays an important and significant role in our daily lives. Mass transfer is a phenomenon that takes place in numerous activities which we undertake during day to day working such as adding sugar to tea, adding salt in a vegetable curry, evaporation of water in to air in a cooler, drying of wet clothes, etc.

Mass transfer involves movement of matter of a substance from one place to another place. It is different from movement of bulk fluid such as air movement caused by a fan or blower and flow of water caused through a pipe due to pressure difference or by a pump. In mass transfer, movement is caused by differences in concentration of the substances between two regions. Mixing of two gases upon removal of a boundary separating them in a container is an example of mass transfer on account of concentration differences as shown in Figure 1.

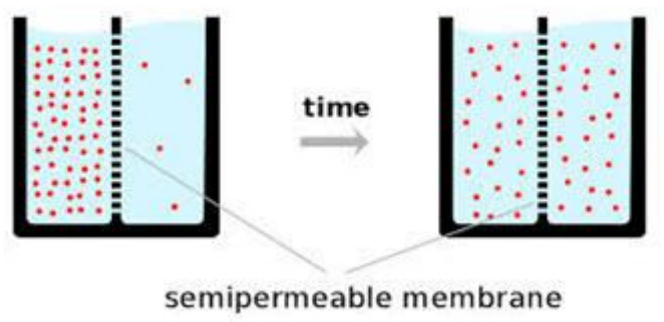



**Figure 1**

Concentration of a substance quantifies the amount of the substance per unit volume. This amount can be on a mass or molar basis. The mass concentration of a particular substance called the density and is expressed as

$$\rho_k = \frac{m_k}{V} \qquad (1)$$

In mass transfer concentration means molar concentration and is expressed as

$$C_k = \frac{n_k}{V} \qquad (2)$$

In mass transfer, movement of a matter of substance occurs due to concentration gradient and movement is always from high concentration region to low concentration region. The mass transfer will continue till the concentration differences between two regions exist and will stop when equilibrium is obtained. Mass transfer basically deals with transport of species:

• **within a medium** for example sugar dissolves in a cup of tea to sweeten the entire tea cup as shown in Figure 2.

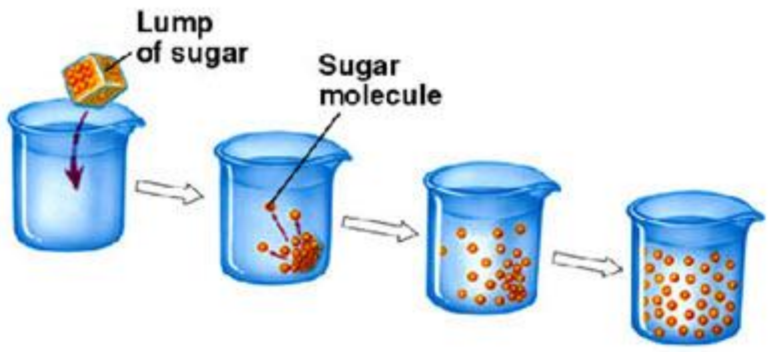



Figure 2

• **across an interface** for example from one medium to another i.e. spreading of food odour in the entire house as shown in Figure 3.

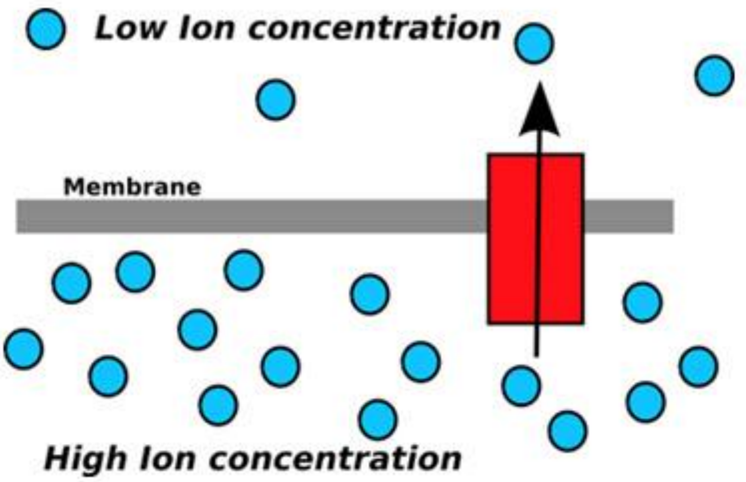



Figure 3

Mass transfer is similar to heat transfer in following ways:

- The driving force for heat transfer is temperature gradient whereas mass transfer occurs due to concentration gradient.
- Heat transfer always takes place from high to low temperature regions, similarly mass is transferred towards low concentration regions, thereby, decreasing the temperature gradient.
- Heat transfer stops immediately when temperature difference becomes zero, similarly, mass transfer ceases when concentration gradient is reduced to zero.
- The rates of heat and mass transfer depend upon the driving potential and resistance.

**Types of Mass Transfer:**

Transfer of mass takes place under different conditions and depending upon the conditions, it can classified in to different categories which are shown in Figure 4.

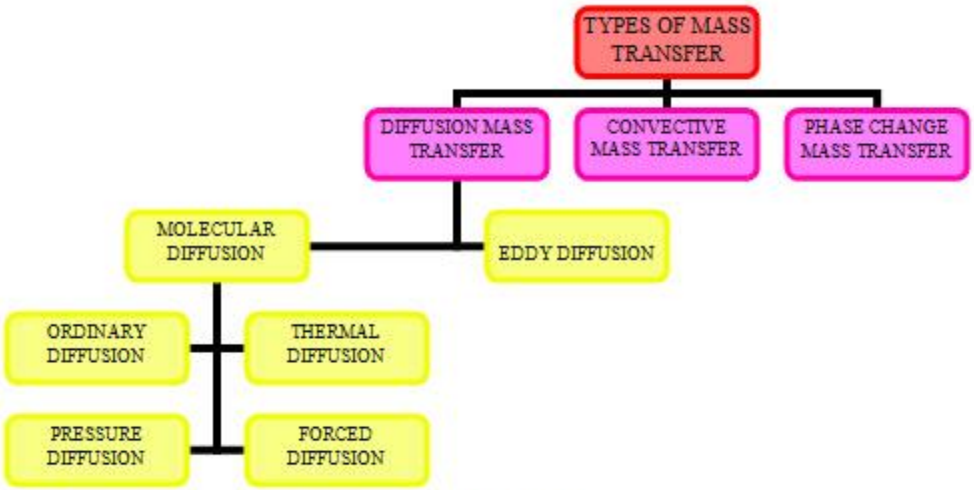



**Figure 4**

**1. Diffusion Mass Transfer:** Diffusion mass transfer can be classified two categories:

i) **Molecular Mass Diffusion**: This type of mass transfer occurs at macroscopic level in which transfer of mass takes place from a region of high concentration to low concentration in a mixture of liquids or gases. Transfer of mass by diffusion occurs due to

- **Presence of concentration gradient** in a mixture and is called ordinary diffusion as shown in Figure 4.

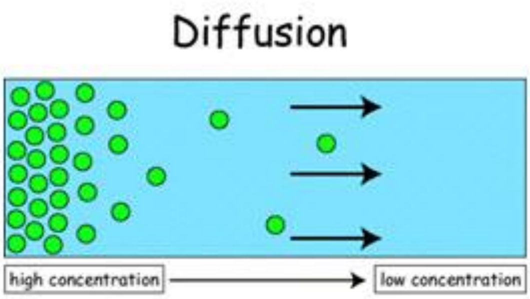



**Figure 4**

- **Presence of thermal gradient** and is termed as thermal diffusion
- **Presence of pressure gradient** and is termed as pressure diffusion
- **Presence of external forces** and is termed as forced diffusion

ii) **Eddy Diffusion:** Mass transfer by eddy diffusion occurs when one of the diffusing fluids is in turbulent motion and is in addition to the diffusion mass transfer. The turbulent motion increases mass transfer.

**2. Convective Mass Transfer:** Mass transfer occurring between a moving fluid and a surface or between two relatively immiscible fluids is termed as convective mass transfer.

**3. Mass Transfer by Phase Change:** This type of mass transfer occurs due to change in the phase of a substance.

**Fick's Law of Diffusion:**

The diffusion process is governed by mass transfer laws which are very similar to heat transfer laws and govern the relationship between mass flux and concentration gradient. The **basic law of diffusion** was proposed in 1855 by **Adolf Fick** which is expressed as

Mass Flux = Constant of Proportionality X Concentration Gradient (1)

Consider a system in which a partition separates two gases B and C and the two gases are maintained at same temperature and pressure initially as shown in Figure 5.

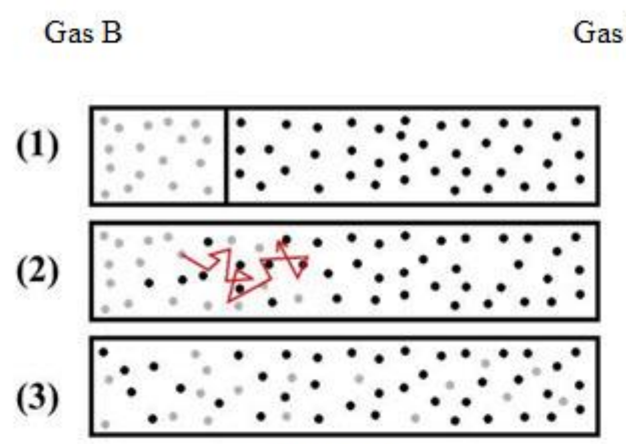



Figure 5

On removal of the partition, molecules of gas B move towards right where concentration is low while molecules of gas C move towards low concentration area i.e towards left. The molecules of both the gases diffuse with each other gradually. The diffusion rate is given by Fick's law and is expressed as

$$\frac{m_b}{A} = -D_{bc}\frac{dC_b}{dx} \qquad (2)$$

$m_b/A$ is **Mass flux or mass flow per unit area**, $Kg_m$ /hr-m$^2$

A is area through which mass is flowing, m$^2$

$C_b$ is **mass concentration** or **molecules per unit volume**, kg/m$^3$

$\frac{dC_b}{dx}$ is **concentration gradient** for gas B

$D_{bc}$ is the **diffusion coefficient or mass diffusivity**, m$^2$/Sec

The unit of diffusion coefficient or mass diffusivity is same as the units of thermal diffusivity and kinematic viscosity which is also called momentum diffusivity. Diffusion coefficients are generally determined experimentally and increase with increase in temperature but decrease with increase in pressure.

Mass flux for gas C can be expressed as

$$\frac{m_c}{A} = -D_{cb}\frac{dC_c}{dx} \qquad (3)$$

By using characteristic gas equation, Fick's Law can be expressed in terms of partial pressure.

$$p_b = \rho_b R_b T = \rho_b \frac{G}{M_b} T \qquad (4)$$

$$\therefore\ \rho_b = p_b \frac{M_b}{GT} \qquad (5)$$

Density represents mass concentration to be used in Fick's Law. Therefore,

$$\therefore\ \rho_b = p_b \frac{M_b}{GT} = C_b \qquad (6)$$

Substituting the value of $C_b$ from equation (6) in to equations (2) and (3), we get

$$\frac{m_b}{A} = -D_{bc}\frac{M_b}{GT}\frac{dp_b}{dx} \qquad (7)$$

$$\frac{m_c}{A} = -D_{cb}\frac{M_c}{GT}\frac{dp_c}{dx} \qquad (8)$$

The above equations hold good for isothermal diffusion only.

The Fick's law of diffusion as given by equations (2) and (3) is similar to Fourier's law of heat conduction and Newton's law of viscosity expressed by equations (9) and (10) respectively.

$$\frac{Q}{A} = -K\frac{dT}{dx} \qquad (9)$$

$$\tau = -\mu\frac{du}{dy} \qquad (10)$$

Analogy between mass, heat and momentum transfer has been shown in Table 1

Table 1:

| Transport Phenomenon | Governing Law | Driving Potential |
|---|---|---|
| Mass Transfer | Fick's Law of Diffusion | Concentration Gradient, $\frac{dC_b}{dx}$ |
| Heat Transfer | Fourier's law of conduction | Temperature Gradient, $\frac{dT}{dx}$ |
| Momentum Transfer | Newton's Law of Viscosity | Velocity Gradient, $\frac{du}{dy}$ |

Some important aspects of Fick's law of diffusion are summarized below;

1. Fick's law is applicable for all matter irrespective of it state: solid, liquid or gas. Mass transfer is inversely proportional to molecular spacing.

2. Similar to heat transfer which occurs in direction of decreasing temperature, mass transfer by diffusion also occurs in the direction of decreasing concentration.

3. Apart from concentration gradient, mass transfer can occur due to presence of temperature gradient, pressure gradient or external force. However, Fick's law gives rate of mass transfer on account of concentration gradient only and effect of other parameters is considered to be small or negligible.

4. Diffusion coefficients in gases are highest, followed by liquids and solids.

5. Diffusion coefficient or mass diffusivity is a function of temperature, pressure and composition of a system. However, for ideal gases and dilute liquids, diffusion coefficient depends on temperature and pressure and is independent of system composition.

**Lesson-32 Reynolds Analogy and Numerical problems**

**Mass Transfer Coefficient**:

Till now in our discussion, concentration gradient was considered to be the driving potential for transfer of mass. However, in practical situations involving fluids, convective mass transfer can not be neglected.

The governing equation for convective mass transfer is similar to the convective heat transfer equation and is expressed as

$$m_b = h_{mc} ( C_{b1} - C_{b2}) \qquad (1)$$

where

$m_b$ is is the **diffused mass** of component 'b'

$h_{mc}$ is **mass transfer coefficient** of component 'b'

$C_{b1}$ and $C_{b2}$ are **mass concentrations** of component 'b'

For steady state, one dimensional diffusion of a fluid across layer of thickness ($X_2$ –$X_1$), mass diffusion can be expressed as

$$m_b = -DA\frac{(C_{b1}-C_{b2})}{(X_2-X_1)} \qquad (2)$$

Comparing equations (11) and (12), we get

$$h_{mc} = \frac{D}{(X_2-X_1)} \qquad (3)$$

Equation (3) represents **mass transfer coefficient** based on **concentration gradient**.

We know that mass flux or flow per unit area is represented as

$$\frac{m_b}{A} = -D_{bx}\frac{M_b}{GT}\frac{dp_b}{dx}\text{, we can write}$$

$$m_b = DA\frac{M_b}{GT}\frac{(p_{b1}-p_{b2})}{(X_2-X_1)} \qquad (4)$$

Using equation (3), equation (4) can be written as

$$m_b = h_{mc}\frac{M_b}{GT}A(p_{b1} - p_{b2}) \tag{5}$$
$$= h_{mp}A(p_{b1} - p_{b2})$$

Where $h_{mp}$ is **mass transfer coefficient** based on **pressure**

$$h_{mp} = h_{mc}\, M_b / GT$$

$$= h_{mc} / RT \tag{6}$$

For diffusion of water vapor through a layer of stagnant air, mass diffusion for water is expressed as

$$m_w = DA\frac{M_w p_t}{GT(X_2 - X_1)} loge \frac{(p_t - p_{w2})}{(p_t - p_{w1})} \tag{7}$$
$$= h_{mp}A(p_{w1} - p_{w2})$$

**Mass transfer coefficient**, $h_{mp}$, based on **pressure difference** can be written as

$$h_{mp} = \frac{Dp_t}{(X_2 - X_1)(p_{w1} - p_{w2})}\frac{M_w}{GT} loge \frac{(p_t - p_{w2})}{(p_t - p_{w1})} \tag{8}$$

**Mass transfer coefficient**, $h_{mc}$, based on **concentration gradient** can be expressed as

$$h_{mc} = \frac{Dp_t}{(X_2 - X_1)(p_{w1} - p_{w2})} loge \frac{(p_t - p_{w2})}{(p_t - p_{w1})} \tag{9}$$

**Reynolds Analogy:**

The Reynolds analogy describes analogous behavior of mass, momentum and heat transfer and it was first recognized by Reynolds. The convective transport of mass, momentum and heat normally occur through a thin boundary layer close to the wall. The equations governing the transport of these quantities are analogous if the pressure gradient is equal to zero and the Prandtl Number (Pr) and Schmidt Number (Sc) are equal to unity. Under these conditions, their non-dimensional convective transport coefficients are related by the equation given below

$$Re\frac{f}{2}=Nu=Sh$$

Or
$$\frac{f}{2}\frac{VL_c}{\nu}=\frac{h_{heat}L_c}{k}=\frac{h_{mass}L_c}{D_{bc}} \quad (10)$$

where

f is friction factor

Re is Reynolds Number = $\frac{VL_c}{\nu}$

Nu is the Nusselt Number representing heat transfer = $\frac{h_{heat}L_c}{k}$

Sh the Sherwood Number representing mass transfer = $\frac{h_{mass}L_c}{D_{bc}}$

Equation (10) is known as the Reynolds analogy, and enables the calculation of heat transfer coefficient if either the friction factor or the mass transfer coefficient is known.

**Example 14.1 Estimate the diffusion coefficient for ammonia in air at 25°C temperature and one atmospheric pressure.**

**For ammonia:**

**Molecular weight =20 and molecular volume = 25.81 cm³/gm mole**

**For air:**

**Molecular weight = 26and molecular volume = 29.89 V cm³/gm mole**

**Solution:**

The diffusion coefficient for binary gaseous mixtures is worked out from the relation:

$$D = 0.0043\frac{T^{3/2}}{p_t\left(V_b^{1/3}+V_c^{1/3}\right)^2}\left(\frac{1}{M_b}+\frac{1}{M_c}\right)^{1/2}$$

Inserting the appropriate values in consistent units

$$D = 0.0043\frac{(273+25)^{3/2}}{1\times(25.81^{1/3}+29.89^{1/3})^2}\left(\frac{1}{20}+\frac{1}{26}\right)^{1/2}$$

$$= 0.0043\times\frac{5144.27}{(2.92+3.07)^2}\times(0.05+0.0385)^{1/2}$$

$$= 0.0043\times\frac{5144.27}{35.88}\times 0.297 = 0.1831\ \mathrm{cm^2/s}$$

**Example 14.2 A rectangular system having steel walls of 8 mm thickness stores gaseous hydrogen at elevated pressure. The molar concentration of hydrogen in the steel at the steel at the inner and outer surfaces of the wall are approximated to be 1.0 kg-mol/m³ and 0.0 kg-mol/m³ respectively. Presuming that the binary diffusion coefficient for hydrogen in steel is 0.2410⁻¹² m²/s, workout the diffusion flux for hydrogen through the steel wall. Point out the assumptions made in the derivation of the relation used by you.**

**Solution:**

The molar diffusion flux of hydrogen (h) through the steel wall (s) is prescribed by the Fick's law

$$\frac{m_h}{A} = \frac{D_{hs}(C_{h1} - C_{h2})}{(x_2 - x_1)}$$

Which has been worked out with the following assumptions:

(i) Steady-state conditions
(ii) One-dimensional species diffusion through a plane wall which is approximately as a stationary medium.
(iii) No chemical reaction of the diffusing substance in the solid wall

Inserting the appropriate data in consistent units:

$$N_h = \frac{m_h}{A} = \frac{0.24\times10^{-12}(1-0)}{0.010} = 2.4 \times 10^{-11} \text{ kg-mol/s-m}^2$$

Since the molecular weight of hydrogen is 2 kg/kg-mol, the mass flux of hydrogen, is:

$$= 2 \times 2.4 \times 10^{-11} = 4.8 \text{ kg/s-m}^2$$

**Example 14.3 A plastic membrane 0.25 mm thick has hydrogen gas maintained at pressures of 2.5 bar and 1 bar on its opposite sides. The binary diffusion coefficient of hydrogen in the plastic is 8.510$^{-8}$ m$^2$/s and the solubility of hydrogen in the membrane is 1.510$^{-3}$kg-mol/ m$^2$-bar. Under uniform temperature conditions of 25°C, workout: (a) molar concentrations of hydrogen at the opposite faces of the membrane, (b) molar and mass diffusion flux of hydrogen through the membrane.**

**Solution:**

The molar concentrations (C), the partial pressures (p) and the solubility (S) of the diffusing gas are related to each other by the expression,

$$C = S\,p$$

Therefore molar concentrations of hydrogen at the opposite faces of the plastic membrane are:

$$C_{h1} = 1.5 \times 10^{-3} \times 2.5 = 3.5 \times 10^{-3} \text{ kg-mol/ m}^3\text{-bar}$$

$$C_{h1} = 1.5 \times 10^{-3} \times 2.5 = 3.5 \times 10^{-3} \text{ kg-mol/ m}^3\text{-bar}$$

(b) The molar diffusion flux of hydrogen through the membrane is worked out from the relation

$$N_h = \frac{m_h}{A} = \frac{D_{hp}(C_{h1}-C_{h2})}{(x_2-x_1)}$$

The subscripts h and p refer to hydrogen and plastic, respectively
Inserting appropriate values in consistent units,

$$N_h = \frac{8.5\times10^{-8}(3.75\times1.5\times10^{-3})}{0.5\times10^{-3}} = 38.25 \times 10^{-8}\text{kg-mol/s m}^2$$

Since the molecular weight of hydrogen is 2 kg/kg-mol, the mass flux of hydrogen is

$$= 2 \times 38.25 \times 10^{-8} = 76.5 \times 10^{-8} \text{ kg/ s-m}^2$$

**Example14.4 Hydrogen gas at 1 bar and 400 k flows through a rubber tubing of 10mm inside radius and 20 mm outside radius. The diffusivity of hydrogen through rubber is stated to be 0.7510$^{-4}$ m$^2$/hr and the solubility of hydrogen is 0.052 m$^3$ of rubber t 1 atmosphere. What would be the diffusion loss of hydrogen per metre length of the rubber tubing? It may be presumed that resistance to diffusion of hydrogen from the outer surface of the tube is negligible.**

**Solution:**

The solubility of hydrogen at the operating pressure of 2 atmosphere is:

$$= 1 \times 0.052 = 0.052\ m^3/m^3 \text{ of rubber}$$

Then from the characteristic gas equation

$$pV = m.RT$$

Where the gas constant R for hydrogen is 4240 j/kg k

$$\therefore 1 \times 10^5 \times 0.052 = m \times 4240 \times 400$$

$$m = \frac{1\times10^5\times0.052}{4240\times400} = 0.003066 \text{kg/m}^3 \text{ of rubber}$$

Therefore, mass concentration of hydrogen at the inner surface of the pipe is 0.01635 kg/m$^3$. We can approximate the mass concentration to be zero at the pipe surface as resistance to diffusion is stated to be negligible at that surface. Thus

$$C_{h1} = 0.31635 \text{ kg/m}^2 \text{ and } C_{h2} = 0.0$$

The diffusion flux through a cylindrical system is given by

$$m = \frac{D(C_{h1}-C_{h2})}{\Delta x} A_m$$

$$\Delta x = (r_2 - r_1) = (20-10)\times10^{-3} = 10\times10^{-3}\text{m}$$

$$A_m = \frac{2\pi l(r_2-r_1)}{\log_e(r_2/r_1)} = \frac{2\pi\times1\times10\times10^{-3}}{\log_e(20/10)} = 9.06\times10^{-2}\text{m}^2$$

$$\therefore m = \frac{0.75\times10^{-4}\times(0.003066-0.0)}{10\times10^{-3}}\times(9.06\times10^{-2}) = 2.20\times10^{-5}\text{kg/hr}$$

**Example14.5 The air pressure in a tyre tube of surface area 1 m$^2$ and wall thickness of 0.01m is approximated to drop from 2 bar in a period of 5-days. The solubility of air in rubber is 0.07m$^3$ of rubber at 1 bar. Estimate the diffusivity of air in rubber at the operating temperature of 200 k if the volume of air in the tube is 0.025m$^3$.**

**Solution:**

Initial mass of air in the tube,

$$m_i = \frac{P_i V}{RT} = \frac{2\times10^5\times0.025}{287\times200} = 0.08711\text{kg}$$

Final mass of air in the tube

$$m_f = \frac{P_f V}{RT} = \frac{1.99\times10^5\times0.025}{287\times200} = 0.08667\text{ kg}$$

Mass of air escaped is equal to

$$m_i - m_f = 0.08711 - 0.086677 = 2.9\times10^{4}\text{kg}$$

Therefore, the mass flux equals

$$\frac{m_a}{A} = \frac{\text{mass escaped}}{\text{timeelapsed}\times\text{area}}$$

$$= \frac{2.9\times10^{-4}}{(5\times24\times3600)\times1} = 0.662\times10^{-9}\text{ kg/ s-m}^2$$

The solubility of air at the mean operating pressure of (2+1.99)/2=1.995 bar is

$$= 1.995\times0.07 = 0.1396\text{ m}^3\text{ of air/m}^3\text{ of rubber}$$

The air which escapes to atmosphere will be at 1 bar pressure and its solubility will remain at 0.07 $m^3$ of rubber.

The corresponding mass concentrations are worked out from the characteristic equation,

$$C_{a1} = \frac{PV}{RT} = \frac{1.995\times10^5\times0.1396}{287\times200} = 0.4852\text{ kg/m}^3$$

$$C_{a2} = \frac{1\times10^5\times0.07}{287\times200} = 0.12195\text{ kg/m}^3$$

The diffusion flux of air through rubber is:

$$\frac{m_a}{A} = \frac{D_a(C_{a1}-C_{a2}}{(x_2-x_1)}$$

$$\therefore 1.342\times10^{-9} = \frac{D_a(0.4852-0.12195)}{0.01}$$

Therefore, diffusivity of air in rubber is

$$D_a = \frac{1.342\times10^{-9}\times0.01}{(0.4852-0.12195)} = 3.67\times10^{-11}\text{ m}^2\text{/s}$$

**Example14.6 A distillation column containing a mixture of benzene and toluene is at a pressure of 1 atmosphere and 50°C temperature. The liquid and vapour phases contain 30 mol% and 45 mol % of benzene. At 50°C temperature, the vapour pressure of toluene is 70kN/m$^2$ and the diffusivity is 510$^{-6}$ m$^2$/s. workout the rate of interchange of benzene and toluene between the liquid and vapour phases if resistance to mass lies in film 0.50 mm thick.**

**Take atmospheric pressure=101 kN/m$^2$ and Universal gas constant G=8.314 kj/kg-mol K.**

**Solution:**

Let subscripts b and t refer to benzene and toluene respectively. At the liquid plane 1, the partial pressure of toluene is

$$p_{t_1} = \text{molar concentration} \times \text{vapour pressure}$$
$$= (1 - 0.3) \times 70 = 49 \text{ kN/m}^2$$

For the equi-molal diffusion, the molar diffusion flux of toluene is;

$$N_t = \frac{m_1}{A} = \frac{D}{GT}\frac{(P_{t_1} - P_{t_2})}{(x_2 - x_1)}$$

$$= \frac{5\times10^{-6}}{8.314\times323} \times \frac{(49-55)}{0.25\times10^{-3}} = -44.68 \times 10^{-6} \text{ kg-mol/ s-m}^2$$

The negative sign indicates that transfer of toluene is from vapour to liquid. Benzene will diffuse in the opposite direction at the same rate.

**Example14.7 Derive an expression for the steady state diffusion of a gas A through another stagnant gas B.**

**Oxygen is diffusing through stagnant carbon monoxide at 0°C and 800 mm of Hg pressure under steady state conditions. The partial pressure of oxygen at two planes 0.6 cm apart is 100 mm of Hg and 20 mm of Hg respectively. Calculate the rate of diffusion of oxygen in gm-moles through cm² area. It may be presumed that:**

**Diffusivity of oxygen in carbon monoxide =0.185 cm²/s**

**Gas constant R=82.06 cm² atm/gm mole K.**

**Solution:**

Let subscripts 0 and c refer to oxygen and carbon monoxide.

Partial pressures of oxygen on the given planes are:

$$p_{0_1} = \frac{100}{800} = 0.125 \text{ atm}$$
$$p_{0_2} = \frac{20}{760} = 0.0263 \text{ atm}$$

Partial pressures of carbon monoxide are:

$$p_{c_1} = 1 - 0.125 = 0.875 \text{ atm}$$
$$p_{c_2} = 1 - 0.0263 = 0.9737 \text{ atm}$$

Log mean partial pressure for non-diffusing carbon monoxide is given by:

$$\text{LMPC} = \frac{p_{c_1} - p_{c_2}}{\log_e(p_{c_1}/p_{c_2})}$$

$$= \frac{0.875-0.9737}{\log_e(0.875/0.9737)} = 2.1264 \text{ atm}$$

The diffusion rate of oxygen is given by,

$$N_{0_2} = \frac{DA}{GT} \times \frac{p_t}{(x_2 - x_1)} \times \left(\frac{p_{0_1} - p_{0_2}}{\text{LMPC}}\right)$$

$$= \frac{0.185\times1}{82.06\times273} \times \frac{1}{0.6}\frac{(0.1125-0.0263)}{2.1264} = 5.579 \times 10^{-7} \text{gm-mol/s}$$

Alternatively

$$N_{0_2} = \frac{DA}{GT} \times \frac{p_t}{(x_2-x_1)} \times \log_e\frac{p_{c_2}}{p_{c_1}}$$

$$= \frac{0.185\times1}{82.06\times273} \times \frac{1}{0.6} \times \log_e\frac{0.9737}{0.875} = 6.388 \times 10^{-7} \text{gm-mol/s}$$

**Example14.8 Estimate the diffusion coefficient of carbon tetrachloride into air from the following data recorded in a Stefan-tube experiment with carbon tetrachloride and oxygen:**

**Diameter of tube and its length above liqid surface: 1 cm and 15 cm respectively**

**Temperature and pressure maintained: 0°C and 760 mm of mercury**

**Evaporation of carbon tetrachloride: 0.03 gm in 12 hour**

**Vapour pressure of carbon tetrachloride: 30 mm of mercury**

**Solution:**

The molecular weights of carbon tetrachloride $M_c$ and oxygen $M_0$ are:

$$M_c=12+4(31.5) = 154 \text{ and } M_0= 32$$

Partial pressures of these elements at bottom and top are:

$$p_{c_1} = 30\text{mm of Hg} \quad ; \quad p_{c_2} = 0$$

$$p_{0_1} = 760 - 30 = 730 \text{ mm of Hg}$$

$$p_{0_2} = 760 \text{ mm of Hg and } p_t = 1\times10^5 \text{ N/m}^2$$

$$\text{Further, } m = \frac{0.03\times10^{-10}}{12\times3600} = 6.9416 \times 10^{-10}\text{kg/s}$$

$$\frac{8.333\times10^{-10}}{154} = 4.507 \times 10^{-12}\text{kg-mol/s}$$

$$A = \frac{\pi}{4}(0.01)^2 = 7.854 \times 10^{-5}\text{m}^2$$

$$(x_2 - x_1) = 15\ cm = 0.15m$$

Inserting these values in the diffusion equation,

$$m = \frac{DA}{GT}\frac{p_t}{(x_2 - x_1)}\log_e\left(\frac{p_{0_2}}{p_{0_1}}\right)$$

$$4.507 \times 10^{-12} = \frac{D\times7.854\times10^{-5}}{8314} \times \frac{10^5}{0.15} \times \log_e\left(\frac{760}{730}\right) = 1.1015 \times 10^{-4}D$$

∴ Diffusion coefficient of carbon tetrachloride in air

$$D = \frac{4.507\times10^{-12}}{1.1015\times10^{-4}} = 4.0917 \times 10^{-8} \text{ m}^2\text{/s}$$

**Example14.9 An open tank, 10 mm in diameter, contains 1 mm deep layer of benzene (Mol wt =78) bottom. The vapour pressure of benzene in the tank is 13.15 kN/m² and its diffusion takes place through a stagnant air film 2.5 thick, At the operating temperature of 20°C, the diffusivity of benzene in the tank is 8.010$^{-6}$ m²/s. If the benzene has a density of 880 kg/m³, calculate the time taken for the entire benzene to evaporate. Take atmospheric pressure as 101.3 kN/m² and neglect any resistance to diffusion of beneze beyond the air film.**

**Solution:**

Let subscripts b and a refer to benzene and air respectively.

Partial pressures of benzene and air at the two levels are:

$p_{b_1} = 13.5$ kN/m$^2$ and $p_{b_2} = 0.0$ kN/m$^2$

$p_{a_1} = p_t - p_{b_1} = 101.3 - 13.5 = 87.8$ kN/m$^2$

$p_{a_2} = p_t - p_{b_2} = 101.3 - 0.0 = 101.3$ kN/m$^2$

The diffusion rate of benzene is given by

$$N_b = \frac{DA}{GT}\frac{p_t}{(x_2 - x_1)}\log_e\left(\frac{p_{a_2}}{p_{a_1}}\right)$$

$$= \frac{8.0\times10^{-6}[\pi/4\times(6)^2]}{8.314\times293}\times\frac{101.3}{0.0025}\times\log_e\frac{101.3}{87.8}$$

$$= 5.378\times10^{-4}\text{kg}-\frac{\text{mol}}{\text{s}} = (5.378\times10^{-4})\times78 = 0.04195\text{kg/s}$$

Mass of benzene which is to be evaporated,

$$= \frac{\pi}{4}(10)^2\times0.001\times880 = 69.083\text{ kg}$$

$\therefore$ Time needed for evaporation$= \frac{69.083}{0.04195} =$ **1646.79s**